Complex $su_q(2)$ dynamical symmetry, limiting cases and one-dimensional potential realisations

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Abstract

Using a complex deformation ($q = e^{is}$) of the Lie algebra $su(2)$ we obtain extensions of the finite-dimensional representations towards infinite-dimensional ones. A generalised q-deformation of $su(2)$, as a Hopf algebra, is introduced and it is proved that some infinite-dimensional representations can be also constructed. The Schrödinger picture of $su_{e^{is}}(2)$ is investigated, using a differential realization, and a large class of equivalent potentials is obtained. Limiting cases of the deformations are also analysed. A connection between the case when q is a root of 1 and the commensurability of the corresponding potentials is found.

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Short title: Complex q-deformation of su(2) and potential realization

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1 Introduction

Quantized universal enveloping algebras (also called q-algebras, q-deformed Lie algebras) have been the subject of numerous recent studies in mathematics and physics. They represent some special deformations ($q \neq 1$) of the universal enveloping algebra of Lie algebras ($q = 1$) [1]. They were first introduced by Sklyanin [2] and by Kulish and Reshetikhin [3], and developed through a Jordan-Schwinger realisation of $su_q(2)$ by Biedenharn [4], Macfarlane [5] and Ng [6], or through $su_q(1,1)$ by Ui and Aizawa [7], Kulish and Damaskinsky [8], Maekawa and Masuta *et al* [9] and Gromov and Man'ko [10]. Since then $su_q(2)$ and $su_q(1,1)$ have been applied in various branches of physics. The importance of these deformations, in one-dimensional problems, establishes its role beyond doubt for many physical problems, like: deformed spin-chain models [11], description of rotation-vibration spectra of deformed or superdeformed nuclei [12-14], molecular spectra [15], as well as two-dimensional $su_q(3)$ [16] and three-dimensional $u_q(6)$ [17], or $su_q(N)$ [18] generalisations.

Such deformations provide much interest in various other contexts, e.g. q-deformed SUSY [19], exact solvable potentials [20], Hamiltonian quantization [21] and fractional anyonic statistics [22]. Consequently they might prove useful in applications to nuclear, molecular physics [23] and scattering theory. The method of equivalent potentials developed in [24] sugests the possibility of a continuous (q-deformation) of some dynamical symmetries, i.e. bound states towards scattering states by deforming the harmonic oscillator potential into a Pöschl-Teller-like potential, by both real and complex q-deformation of the energy levels. We also note that q-deformation **with complex q** have been investigated and proved to be usefull in many physical applications [9,10,13,17,22].

Recently, new generalised forms of deformation of Lie algebras have been introduced: deformations involving J_0 only in one commutator [25,26] (the latter reference proving a generalised deformation endowed with a Hopf algebraic structure), or in all theree commutators [27].

A possible procedure, inspired by the generalization of the Algebraic Scattering Theory (AST) into q-deformations, could be: to q-deform a given symmetry with phys-

ical applications, and to smoothly modify the q parameter, in such a way, to find, for certain limiting values, new symmetries of physical interest. More than this, one can investigate what is happening in between these limiting cases, and the possible underlying physics (and some physical significance of the parameter q).

In this paper we add to this field and prove that some certain q-deformations (trigonometric in Sect.2 or more general, in Sect.3) can carry the bounded unirreps of $su(2)$ to unbounded unitary representations, some of them being equivalent to those of $su(1,1)$. We also prove the equivalence of $su_q(2)$ with the two-dimensional Euclidean Lie algebra $e(2)$ for certain values of q. In Sect.4 we find a differential realisation of $su_q(2)$ which can be related to a Schrödinger equation coresponding to some particular one-dimensional potentials of physical interest. These results prove that structures more general then Lie algebras could succesfully be applied to algebraic scattering theory (AST) [28].

The aim of this paper is to implement the q-deformation in the traditional method used in AST, i.e. starting from systems with bound state solutions, described by a compact algebra, and continuosly deforming in order to obtain scattering solutions. In traditional AST this is achived by analytic continuation into the complex plane of the generators (the compact Lie algebra is analytically continued into a non compact one by imposing supplementary conditions on the generators). In the present paper we prove that this could be obtained by using the q-deformation of the compact $su(2)$ Lie algebra to $su_q(2)$, $q = e^{is}$, the later having infinite dimensional unitary representations. In this way the transition from bounded to scattering states in one-dimensional systems could be described in a unified algebraic formalism. More than these, a possible way to get the Euclidean connection [28] through q-deformation, by starting from the compact Lie algebra $su(2)$ and obtain a q-deformed algebra similar with $e(2)$ is worked out in Sect.2. This algebraic result was also realised in a potential picture with q-differential operators for the generators. A large set of potential shapes were obtained from the equivalent Schrödinger equation, for different values of q. We have found a similarity between the rational/irrational character of s/π and the random/incommensurate character of certain exact integrable potentials.

The other possibility, we have found, is to work with $q \in R$ and to modify the defining function $[2J_z]$. Such an example is worked out in Sect.3, where the non-linear

algebra has also a Hopf algebra structure [1,29-32]. This could be usefull in further applications in the quantum mechanics algebraic approach of many particles systems and lattice models.

2 Unitary representations of $su_{q=e^{is}}(2)$

The purpose of this section is to give a brief discussion of some complex q-deformations of the Lie algebra $su(2)$ together with some of its unitary representations. In the following we review some results concerning the q-algebra $su_q(2)$, q generic, [4-5,9,10,29,30]. In the $su_q(2)$, generated by the operators J_+, J_- and J_z, one has the following commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \tag{1}$$

$$[J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}} = [2J_z]_q = [2J_z], \tag{2}$$

where $J_\pm = J_x \pm iJ_y$ (J_x, J_y and J_z self-adjoint operators). The structure given by eqs.(1,2) can be endowed with a real Hopf algebra structure [1,29-30].

The Casimir operator C is given by

$$C = \left[J_z \pm \frac{1}{2}\right]^2 + J_\mp J_\pm = \frac{[2]}{2}[J_z]^2 + \frac{1}{2}(J_+J_- + J_-J_+) + \left[\frac{1}{2}\right]^2. \tag{3}$$

For $q \in R$ the unirreps of $su_q(2)$, all finite-dimensional, acting on a Hilbert space V_j with the basis $|jm>$, are characterized by the relations (e.g. [5])

$$C|jm> = \left[j + \frac{1}{2}\right]^2 |jm>, \tag{4}$$

$$J_z|jm> = m|jm>, \tag{5}$$

$$J_\pm|jm> = ([j \mp m][j \pm m + 1])^{1/2}|jm \pm 1>, \tag{6}$$

where m takes integers or half-integers ranging from $-j$ to j for each $j = 0, \frac{1}{2}, 1, ...$ These representations are in one-to-one relation with the unirreps of the Lie algebra

$su(2)$ and coincide with them in the limit $q \to 1$. Infinite dimensional representations for the quantum algebra $su_q(1,1)$ can be obtained in a similar way. The commutators in eq.(1) remain unchanged and eq.(2) becomes

$$[J_+, J_-] \;\; = \;\; -[2J_z]. \tag{7}$$

One obtains the four classes of unirreps for $su_q(1,1)$ [9,11, 21, 29-31] (the principal and complementary series for continuous variation of j and positive and negative discrete series for discrete j) labeled by j, through

$$C|jm> \;\; = \;\; [j][j \pm 1]|jm> . \tag{8}$$

The unirreps reduce also to those of $su(1,1)$, in the limit $q \to 1$. We mention that several excellent papers about this topics exist (i.e. unirreps of $su_q(2)$ and $su_q(1,1)$, or, in general $su_q(n)$ or $su_q(n,m)$, [8-10,21,29-31], and for a recent review see [30] and references herein, which analyse and clasify their unirreps. Here, we only want to give a short review and pictural examples which can exemplify the status of the problem and could be used to give an easier way of understanding and using these techniques.

It is already a classical result [1,5-7,21,29,30-32] that, for real q, the unirreps of $su_q(2)$ are equivalent with those of $su(2)$ (Lusztig and Rosso's Theorem, see for details [29,30]). This fact is remarkably explained by Biedenharn [29] as being the consequence of the invariance of the (discrete) integer number which labels the dimension of the unirrep, versus the continuous variation of q, especially in the case of simple compact Lie groups. In the case when q is a root of 1, the representation theory becomes strikingly different due to the fact that the generators and the Casimir operator become nilpotent elements. The representations are indecomposable and no longer irreductible in general [29,30]. These representations can be clasified, acording to their q-dimension $d_q = \sum_{rep.} q^{J_z}$ into two classes: with $d_q = 0$ and with $d_q \neq 0$.

In the following we show that infinite-dimensional unitary representations can also be obtained directly from $su_q(2)$ with $q = e^{is}$. We extend the q-deformed algebra, $su_q(2)$, eqs.(1,2), to a complex q-deformed algebra, by imposing $q = e^{is}$, $s \in (0, \pi)$ (i.e. $|q| = 1$) with $s \neq k\pi$, $k \in N$. We discuss the cases $s \to 0$, $\pi/2$ and π separately. This algebra is not homeomorphic with $su_q(1,1)$ due to eq.(2) but some similarities with the unbounded representations of $su_q(1,1)$ and $sl_q(2)$, [33], are existing. The equivalence

between $su_{-1}(2)$ with $su(1,1)$, already studied in [34] and noted in the conclusions of [22] represents the basis for our following calculations.

In this Section $q = e^{is}$ and

$$[x]_{q=e^{is}} = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{sin(xs)}{sin(s)} \equiv [x], \tag{9}$$

for a quantity x (a number or operator) will be used without any label, for $q = e^{is}$. When x is a real number, $[x]$ is also real. We note that for $q = e^{is}$ the sign of $[x]$ is not necesary the same with the sign of x, like it is always in the real q-deformation case, $q = e^s$. In addition, for any sequence $\{x_n\}_{n \in Z}$, the q-deformed one $\{[x]\}_{n \in Z}$ is always bounded. These are the points which directed our studies to investigate the $su_q(2) \simeq su(1,1)$ similarity. For example, for a complex number $x = \alpha + i\beta$ we have

$$[x] = [\alpha]\sqrt{1 + [\beta]^2_{e^s}\Big(1 - \frac{[2]^2_{e^s}}{4}\Big)} + [i][\beta]_{e^s}\sqrt{1 - [\alpha]^2\Big(1 - \frac{[2]^2}{4}\Big)}. \tag{10}$$

The object $[x]$ is invariant under the transformation $q = e^{is} \to q^{-1} = q^*$ or, equivalentely, $s \to -s$.

For unitary representations we need the conditions $J_\pm^\dagger = J_\mp$ and $J_z^\dagger = J_z$ which imply that the eigenvalues of J_+J_- and J_-J_+ must be positive and those of J_z must be real. By computing the matrix elements of J_+ and by using eq.(2) it results that two consecutive values of m must differ only by ± 1, like in the classic case of $su(2)$. The operators J_z and C (eq.(3)) form a complete set of commuting self-adjoint operators, $C|cm >= c|cm >$, $J_z|cm >= m|cm >$, $m \in R$ and $c \geq 0$.

As $|cm >$ is eigenvector for $J_\pm J_\mp$ we have

$$J_\pm J_\mp |cm > = \Big(c - \Big[m \mp \frac{1}{2}\Big]^2\Big)|cm > . \tag{11}$$

From eq.(11), the unitarity condition is fulfiled if (see for example [35]):

$$c \cdot \sin^2 s \geq \sin^2 s\Big(m \mp \frac{1}{2}\Big). \tag{12}$$

We consider now the unitary representations of $su_{e^{is}}(2)$, with $s \neq 0, \pi$ to be unitary extensions of the unirreps of $sl_q(2)$. In both these limits $s \to 0$, π, eq.(12) gives the finite-dimensional unirreps of $su(2)$. For $s \to 0$, m is integer or half-integer but for the limiting case $s \to \pi$, m is only integer. In the following we want to present some

of the unitary representations of $su_{e^{is}}(2)$ according to the general procedure used in AST references. By this we mean that we follow the criterium of unitarity, given by eq.(12), regarding the algebra defined with eqs.(1,2) with $q = e^{is}$. We take here into account the algebraic structure, ignoring the Hopf algebraic structure. This will be analysed, for other deformations, in Section 3. We classify the unitary representations by the different admissible ranges of c. We do not take into account other existing equivalent representations, like , for example, the one generated by the automorphism of $su_{e^{is}}(2)$: $J_+ \to -J_+$, $J_z \to J_z + \frac{i\pi}{2}$, without a classical counterpart [1,9,10,30-32]. This representation can be obtained in our case by a shift of π in s.

We investigate such unitary representations, which fall into one of the following three categories:

1. Infinite dimensional continuous series, Fig.1a

(c continuous, bounded from below, m unbounded)

We can take $s \in (0, \pi)$ since, for the other half of the interval, $(\pi, 2\pi)$, the calculations are similar. We label the unirreps by the c and m numbers. Following the inequality in eq.(12), this class, (1), is defined for c within the range: $\frac{1}{sin^2(s)} < c < \infty$ (having continuous variation bounded from below) and m taking all the integers or half-integers (unbounded form above and below) with the restrictions $m_k \neq \frac{\pi}{2s} \pm \frac{1}{2} + \frac{k\pi}{s}$, ($m = m_k$ only in the limit $s = \pi/2$). We can also label these unitary representations by corresponding values for $j = \frac{\pi}{s}\left(k + \frac{1}{2}\right) - \frac{1}{2} + i\sigma$, $\sigma \in R$, $k \in Z$, i.e.:

$$c = \left[j + \frac{1}{2}\right]^2 \;=\; \frac{cosh^2 s\sigma}{\sin^2 s} > \frac{1}{sin^2 s}. \tag{13}$$

One can see that the expression of the label j is not uniquely determined by σ, hence we have obtained the same uncertainty regarding the labeling of these representations, like in the case when q is a root of 1, [29-30]. In these series the minimum value of the eigenvalue of C coincides with the extreme values of the Casimir operator of the "boundary" representations of $sl_q(2)$ (q a root of 1) obtained by Alexeev *et al*, (eq.(18) in [33]). These series are also similar with the principal series in [9] for $su_q(1,1)$ or with those belonging to the quantum Cayley-Klein algebras $su_q(2;j_1)$ in [10] for $j_1 = i$ ("the

strange series"). The action of $J_{\pm}$ gives

$$J_{\pm}|c,m>= N^{\pm}_{cm}|c,m>$$

$$= \frac{(\cos^2\beta_{\pm}\cosh^2(\sigma s)+\sin^2\beta_{\pm}\sinh^2(\sigma s))^{1/2}}{\sin s}|c,m\pm 1>, \tag{14}$$

with $\beta_{\pm} = \mp ms - \frac{s}{2}$. The coefficients $N^{\pm}_{cm}$ are real and become zero when $\sigma \to 0$ and $m = \pm\left(k+\frac{1}{2}\right)\frac{\pi}{s} \pm \frac{1}{2}$. When $\sigma \to 0$ we obtain the limiting value for the Casimir $c \to \frac{1}{\sin^2 s}$. Consequently, we have shown that the representations of this class are unitary and infinite-dimensional, according to eqs.(12,14). Since $N^{\pm}_{cm} \neq 0$, it results that $J_{\pm}$ do not act nilpotently on the space of representation and this becomes irreducible. An example of such series is ploted in Fig.1a, with full circles, in the range $c > c_0$.

2. Mixed series, infinite and finite-dimensional unirreps, Fig.1a.

(c continuous, bounded from below and above, m bounded or unbounded)

The alowed ranges of c and m for these series are obtained again from eq.(12). Due to the trigonometric functions involved in this inequality, the allowed values for m are in certain periodically distributed intervals, which we describe below. In this case the Casimir eigenvalues take values in the range $c_1 = \frac{1}{4\sin^2(s/2)} < c \leq c_0 = \frac{1}{\sin^2 s}$. The allowed range for m is given by

$$\bar{J}_\delta \cup \bar{J}_\Delta = \left[\frac{\pi}{2s}+k\frac{\pi}{s}-\frac{\delta}{2}, \frac{\pi}{2s}+k\frac{\pi}{s}+\frac{\delta}{2}\right] \cup \left[\frac{\pi}{s}+k\frac{\pi}{s}-\frac{\Delta}{2}, \frac{\pi}{s}+k\frac{\pi}{s}+\frac{\Delta}{2}\right] \tag{15}$$

with $k \in \mathbf{Z}$ and J_δ, J_Δ being open intervals on the m-axis, of lenghts $\delta = \frac{s-\pi+2\alpha}{s}$, $\Delta = \frac{2\alpha - s}{s}$, respectively, with $\alpha = \arcsin\sqrt{c\sin^2 s}$. We have denoted with $\bar{J}_\delta, \bar{J}_\Delta$ the closure of these intervals. These series allow the α, δ and Δ parameters to take values in the intervals: $\alpha \in \left[\frac{\pi-s}{2}, \frac{\pi}{2}\right]$, $\delta \in [0,1]$ and $\Delta \in \left[\frac{\pi-2s}{s}, \frac{\pi-s}{s}\right]$, function of the eigenvalue c. In between the allowed intervals for m, there are forbiden intervals J_f of equal lenght $\frac{\pi-2\alpha}{s} \in [0,1]$. All these sequences $(J_\delta, J_f, J_\Delta, J_f)$ have periodicity $\frac{\pi}{s}$. These series can be divided into three different sub-classes:

a. Infinite dimensional unitary representations

(c continuous and bounded, m unbounded)

We obtain infinite dimensional unitary representations with bounded values for c and unbounded for m when $s = \frac{\pi}{k+1}$; $k \in \mathbf{N}, k \neq 0$, (the so called roots of 1). Here $m = m_0 + l$, $l \in \mathbf{Z}$ and we choose m_0 such that should be contained in one of the J_δ allowed intervals. This does not change the generality. $m_0 = \frac{\pi}{2s} - \frac{\delta}{2} + \epsilon$ with ϵ arbitrary within $\epsilon \in \left(0, (k+1)\frac{2\alpha}{\pi} - k\right) \in (0,1)$ such that $m \in \bigcup_k (J_\delta \cup J_\Delta)$ only. Here k fixes the number of states in each of the intervals J_Δ and $\Delta \in (k-1, k)$. In the interval δ we have always one single state. We note that these series $\underline{(2a)}$, labeled by c and ϵ, are similar with the complementary series of $su(1,1)$, because the eigenvalue of the Casimir takes (continuously) any value in the interval $(c_1, c_0) = \left(\frac{1}{4\sin^2(s/2)}, \frac{1}{\sin^2 s}\right)$. A peculiarity of these series is that the values of the numbers $N_{\pm}^{cm}$ have exact only $N+1$ distinct values, which repeat after each sequence of states of the form $|c, m>, ...|c, m+k+1>$, or, in terms of intervals, repeats after each $J_\delta \cup J_\Delta$ sequence. Such an example is plotted with full dots in Fig.1a, for $k = 1$, in the range $c_1 < c < c_0$.

Another set of infinite-dimesional unitary representations can be constructed for $s = \frac{n}{k\pi}$, $n, k \in \mathbf{Z}, n < k$, for the same range of c. The procedure is identical with the previous one with the difference that the periodical structure of the m's will cover more than one $J_\delta \cup J_\Delta$ sequence. This fact is easy to prove if we project the periodical sequence $J_\delta, J_f, J_\Delta, J_f$ on a circle of radius $1/2s$ and find that always exists an integer number N such that for an appropriate m_0 the label $m_0 + j$, $j = 0, 1, ...N$ generates a periodical sequence on this circle. Since N is finite we can always find a set of m_0, s and c such that the forbiden intervals do not contain any state of this sequence. When q is, in general, a root of 1 such representations are called "cyclic" [29,30,32,33].

We do not analyse in detail the situation when $s = \xi\pi$ with ξ irrational. Since $\left[m+\frac{1}{2}\right]$ is randomly distributed in [0,1] we have for $c > c_0$ the continuous representations of class $\underline{(1)}$. The period of m (i.e.1)and of $\left[m \pm \frac{1}{2}\right]$ are incommensurate. We only mention that in the case of the upper limit, $c = \frac{1}{\sin^2 s}$ we obtain representations with a special behaviour in the limit $m \to \infty$, ("strange series"). In order to have infinite-dimensional unitary representations, we need, for any m, the condition $N_{\pm}^{cm} \neq 0$, i.e. $m \neq m_f = \frac{\pi}{s}\left(k + \frac{1}{2}\right) \pm \frac{1}{2}$. We can write the allowed values of m like $m_l = m_0 + l$, $l \in Z$, m_0 arbitrary. Due to the irrationality of s/π we can never have $m_l = m_f$, for any finite l and k. However, in the limit $l, k \to \infty$ the sequences of m_l and m_f could have the

same "accumulation point" in the sens that for some sub-sequences of m_l and m_f we have $\lim_{l,k\to\infty} \frac{m_l}{m_f} = 1$. This gives the "strange" behaviour of such series at infinity and it is an aspect which could be correlated with the problem of random potential theory.

b. Finite dimensional unitary representations of dimension N+1.

(c bounded and discrete, m bounded)

These unitary representations exist for any $s \in (0, \pi)$. The values of m are only in the $\bar{J}_\Delta$ domains and $m = m_0, m_0+1, ..., m_0+N$. The value m_0, which fixes the begining of each series, is given by: $m_0 = \frac{\pi}{s} - \frac{\Delta}{2} + \frac{k\pi}{s}$ with $k \in \mathbf{Z}$, i.e. different non-equivalent representations for different k's. The Casimir eigenvalues are $c = \left[\frac{N+1}{2}\right]^2$ with N given by the condition $\cos^2 \frac{s}{2} < \sin^2 s\left(\frac{N+1}{2}\right)$. An example for $N = 4$ is plotted in Fig.1a, with stars.

c. Finite dimensional unitary representations of dimension 1.

These representations belong to the situations when m is only in the J_δ intervals and $c = c_2 = \frac{1}{4\sin^2(s/2)}$. In this case $m = m_0$ only, with $m_0 = \frac{\pi}{s} + \frac{k\pi}{s}$, $k \in \mathbf{Z}, \delta = 0$ and s takes any value in $(0, \pi)$.

3. Discrete series, only finite-dimensional unitary representations, Fig.1b

(c discrete, bounded and m bounded)

This is the case when $c_2 = \frac{1}{4\cos^2(s/2)} < c < c_1 = \frac{1}{4\sin^2(s/2)}$, and only finite dimensional series exist, due to the fact that the forbiden intervals for m are larger than 1. This case is valid for $s \in (0, \pi/2)$. At $s = \pi/2$ the allowed domain vanishes and for $s \in (\pi/2, \pi)$ we obtain the same numbers as for $s \in (0, \pi/2)$ but the ends of the allowed interval for m interchange their expressions due to the symmetry of the equations. The dimension of each unitary representation in this class is given by $N \in \left(0, \frac{\pi}{s} - 2\right)$, depending on the c value in this range. We have $m \in \bar{J}_\Delta = \left[\frac{\pi}{s} + k\frac{\pi}{s} - \frac{\Delta}{2}, \frac{\pi}{s} + k\frac{\pi}{s} + \frac{\Delta}{2}\right]$. In each of these cases we can extra-label the number m by $m^{(l)} = \frac{\pi}{s} + k\frac{\pi}{s} - \frac{\Delta}{2} + l$ for $l = 0, 1, ..., N$ ($k \in \mathbf{Z}$ labels the representation). We have also $c = \left[\frac{N+1}{2}\right]^2$ or equivalently $c = \left[j + \frac{1}{2}\right]^2$ if we denote $N/2 = j$ and consequently the dimension of the unitary representations are $N + 1 = 2j + 1$ with $N_{max} = \frac{\pi}{s_{min}} - 2$, similar with the results presented in [33]. An example with $N = 5$ is ploted in Fig.1b.

Comments and examples

The above presented clasification is in agreement with the traditional clasification in the existing literature, i.e.: spliting of the unitary representations in the heighest weight (of type (2b), (2c) and (3) in our case) and cyclic ones (of type (1) and (2a)), [29,30].

We should like to do some comments and to workout some simple examples concerning some special values for s and c in these representations. First we mention that for the last interval of the eigenvalue $c < \frac{1}{\cos^2(s/2)}$ there are no unirreps. Second, we mention that the mixed series (2) and the discrete series (3) coincide with those given in [30,33] for the case when q is a root of the unity.

For both the mixed (2) and discrete (3) series presented above, we have the action

$$J_{\pm}|c,m> \;=\; \sqrt{c-\left[m\pm\frac{1}{2}\right]^2}\,|c,m\pm 1>, \tag{16}$$

according to the corresponding unirreps in [5], up to an undetermined phase factor.

We note that though we have deformed a compact Lie algebra $su(2)$, the above q-deformation allows the existence of continuous series for $su_q(2)$, equivalent with C_x^0 principal series of $su(1,1)$, up to a redefinition of the Casimir, $C \rightarrow C + \frac{1}{4} - \frac{1}{\sin^2 s}$. Different from the notations of the above cited papers, we label the representations with the c number instead of j. We obtained continuous series, not only for the $s \rightarrow \pi$ $(q=-1)$ case like in [34] but also for other values of the deformation parameter s (class (1) and (2a)).

We want to note also the existence of some special deformations. In the limit $s \rightarrow \pi$ we have

$$[2J_z]|cm> \;=\; \begin{cases} -2m|cm> & m \in \mathbf{Z} \\ 2m|cm> & m \in \mathbf{Z}+\frac{1}{2} \\ \infty & \text{in the rest} \end{cases} . \tag{17}$$

In this last case we can explain what we mean by the similarity between $su_{e^{i\pi}}(2)$ and $su(1,1)$, for $m \in Z$, through eq.(7). For $s \rightarrow \pi$ and $m \in Z$ the unitary representations (2a) of $su_{e^{is}}(2)$ are equivalent with the C_x^0 unirreps of $su(1,1)$: in both these representations m has the same range and the commutators are identical on the basis $|cZ>$. The single apparent problem consist in the fact that the Casimir operator has poles.

In order to obtain the equivalence we can introduce a manifold of smooth functions $\sigma : [0, \pi] \to R$ such that $\sigma(\pi) = 0$, $\left|\frac{d\sigma}{ds}\right|(\pi) < \infty$. This manifold is continuous with respect to a certain parameter which can label these functions, e.g. $\alpha = \frac{d\sigma}{ds}(s = \pi)$. In this way, for any s we can choose a certain element of this class, $\sigma_\alpha(s)$, such that $|j(\sigma)m >= |\alpha m >$ and the functions $\sigma_\alpha(s)$ are choosen such that the limit $s \to \pi$ exists for C on this basis.

For $s \to \pi/(k+1)$, $k \in \mathbf{N}$, $k \neq 0$, the action of $[2J_z]$ is

$$[2J_z]|c, m >= \frac{\sin \frac{(2\tilde{m}+1)\pi}{(k+1)}}{\sin \frac{\pi}{k+1}} |c, m \pm 1 >,$$

where $\tilde{m} \equiv m_{(k)}$ is m modulo k. Consequently there are only k distinct eigenvalues for $[2J_z]$, and also for $J_\pm$. The states $|c, m >_{s=\pi/(k+1)}$ have infinite degeneration for each value of m. We present graphicaly such an example in Fig.2, where the first 9 eigenvalues of $[2J_z]$ are ploted against s. In the case when $s = \pi/2$ we have

$$[2J_z]|cm >= \begin{cases} \pm 1|cm > & m \in \mathbf{Z} + \frac{1}{2} \quad \text{(quasi-spin-like representations)} \\ 0 & m \in \mathbf{Z} \quad \text{(e(2)-like commutators)} \end{cases}$$

We note that in the second case of the above relation, the commutators of the resulting $su_i(2)$ algebra, i.e. eq.(1) and

$$[J_+, J_-] = 0, \tag{18}$$

provide an algebra homeomorphism with the Euclidean Lie algebra $e(2)$, [35], on the subspace of the unitary representations $|c, \mathbf{Z} >$ only. We mention that previously we have used the word "similarity" instead of algebra homeomorphism, because in the q-deformed cases the action of $[2J_z]$ is identical with the action of $2J_z$, only on same certain subspaces of the representations, for the corresponding undeformed Lie algebras (i.e. $su(1,1)$ and $e(2)$ in the above examples). Evidently a homeomorphism involving $2J_z$ and its q-deformation could be established at the level of the universal covering algebra, but such an analysis is beyound the aim of this paper. Other examples of such homeomorphisms between $su_{e^s}(2)$ and other non-linear algebras are investigated in [25,27]. More, as a support of the above comments, we can reobtain, in our case, the Maekawa's equations for the Casimir, [9], for $q = e^{2is}$

$$C = \cos s[J_z]^2 + \frac{1}{2}(J_+J_- + J_-J_+) \tag{19}$$

$$= \cos s[J_z]^2 + J_x^2 + J_y^2 + \frac{1}{4\cos^2(s/2)}.$$

We note that in the limit $s \to \pi/2$ we obtain

$$C_{s=\pi/2} \quad \to \quad J_x^2 + J_y^2 + \frac{1}{2}, \tag{20}$$

which gives the Casimir of the $e(2)$ Euclidean Lie algebra, in agreement with the consequences of eq.(18) (see also [36] for $e_q(2)$). In the case $s \to \pi$ we obtain from eq.(19)

$$C = -J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+),$$

but again only on the sub-space of states $|c, \mathbf{Z}>$, which, together with the first line in the RHS of eq.(17) and with the comments in the construction of the unirreps of class $\underline{(1)}$ and $\underline{(2)}$, support the asumption for the similarity of $su_{e^{i\pi}}(2)$ with $su(1,1)$. This is in agreement with the conclusions in the references [22,34], too. We would like to note that such a behaviour could be a good start for an analysis of the transition from bounded to unbounded states of a system having q-dynamical symmetry, in the AST frame. These features can provide also a new possibility for the introduction of the Euclidean connection, as a natural extension of the initial undeformed symmetries, [19,20,28,34,36].

However we would like to mention another consequence of eq.(19), which supports the comparison between $su_{e^{is}}(2)$ and $su(1,1)$. According to the geometrical interpretation of Alhassid *et al* in the second reference in [28], the surfaces of constant eigenvalues of the Casimir operator, described in a formal 3-dimensional vectorial space of (J_x, J_y, J_z) , have (for $su_{e^{is}}(2)$) the form

$$C \quad = \quad \cos s \frac{\sin^2(sJ_z)}{\sin^2 s} + J_x^2 + J_y^2 = const. \tag{21}$$

Loosely speaking, these formal "surfaces" are a sort of deformations of the sphere associated with the geometry of the Lie group $SU(2)$. For c and s in the range of the infinite dimensional unirreps $\underline{(2a)}$, they form an infinite countable disconnected set of identical closed sufaces. In the infinite dimensional unirreps case $\underline{(1)}$, with the increasing of s, these surfaces become connected together into one single connected surface, i.e. an open surface, homotopic with the hyperboloid surface associated with the non-compact Lie group $SU(1,1)$, [28]. Of course, in our approach one can not

give a rigurous group theoretical or algebraic interpretation to these algebraic surfaces because, in the q-deformed case, there is no associated Lie group manifold, or at least in this approach. We illustrate a two-dimensional section (i.e. in the (J_z, J_x)-plane) of such "surfaces", for the compact and non-compact limiting cases in Fig.3. In other words the topology of these "surfaces" is controled by the parameter s (i.e. q) when ranges from 0 to π.

3 Generalised real deformation for $su(2)$ as a Hopf algebra

In this Section we intend to find other connections of $su(2)$ with different nonlinear algebras and to generalise the exponential deformation of $su_q(2)$ to a polynomial one and to find a possible differential realisation of it. The next possibility which we sugest for deforming the bounded unitary representations of $su(2)$ into infinite-dimensional ones is to keep the deformation parameter $q \in \mathbf{R}$ and to generalise the deformation function, i.e. the RHS of eq.(2). Several succesfull attempts were made in these direction [25-27,37,38] and new kinds of nonlinear algebras, together with their unirreps were constructed (most of them having exponential-like spectra for $[2J_z]$). In the following we apply the deformation introduced by Ludu et al. in [26], mainly for the fact that it can be endowed with a real-hopf algebra structure.

Starting from $su(2)$ we construct a deformed (nonlinear) algebra $L_{g,q} \in U(su(2))$ (the universal covering algebra), q real, generated by the operators $J_z = J_z^\dagger$, J_+, $J_- = J_+^\dagger$, the identity I, g and g^{-1}. The last two elements can also be considered as two holomorphic real functions $g(J_z, q), g^{-1}(J_z, q)$ which belong to the polynomial ring $C[J_z, J_z^{-1}]$ and are defined by the relations

$$gg^{-1} = g^{-1}g = 1, \tag{22}$$

$$g^2 J_\pm g^{-2} = q^\pm J_\pm, \tag{23}$$

$$[J_+, J_-] = 2\frac{g^2 - g^{-2}}{h(q)}, \tag{24}$$

where $h(q)$ is an analytical function of q. It is proved [26] that $L_{g,q}$ satisfies the postulates of the Hopf algebra [29-32] if

$$\Delta(J_\pm) = J_\pm \otimes g^{-1} + g \otimes J_\pm, \tag{25}$$

$$\Delta(g^{\pm 1}) = g^{\pm 1} \otimes g^{\pm 1}, \tag{26}$$

$$\epsilon(J_\pm) = 0, \qquad \epsilon(g^{\pm 1}) = 1, \tag{27}$$

$$S(J_\pm) = -q^{\mp 1} J_\pm, \tag{28}$$

$$S(g^{\pm 1}) = g^{\mp 1}. \tag{29}$$

The algebra $L_{f,q}$reduces to $su(2)$ in the limit $q \to 1$ if the following conditions are fulfilled

$$f(J_z, 1) = 1, \tag{30}$$

$$\frac{\partial f}{\partial q}(J_z, 1) = J_z, \tag{31}$$

$$h(q) = q - q^{-1}, \tag{32}$$

where we denote $g^{\pm 2} = f^{\pm 1}$. If $f = q^{J_z}$ then $L_{g,q}$ is homeomorphic with $sl_q(2, C)$ for complex q and with $su_q(2)$ for real q [1,30]. In this latter case the q-deformed algebra has a real-Hopf algebra structure defined by the relations (22-32).

The Casimir operator of $L_{g,q}$ is given by [26,27,30]

$$C = 2^{1/3}\Big[\frac{2(q^{-1/4}g - q^{1/4}g^{-1})^2}{(q-q^{-1})^{4/3}(q^{1/2}-q^{-1/2})^{2/3}} + \frac{J_+J_-}{(q^{1/2}+q^{-1/2})^{1/3}}\Big]. \tag{33}$$

In order to obtain infinite-dimensional representations for $\mathcal{L}_{g,q}$ we introduce the definition

$$f(J_z, q) = 1 + (q-1)J_z + (q-1)^2(\alpha J_z + b(J_z)), \tag{34}$$

where $b(J_z)$ is a positive, bounded analitical function, arbitrary for the moment. It is easy to verify that the function (34) fulfills the conditions (30-31). For $q = q_1 = \frac{\alpha-1}{\alpha}$ we have $f(J_z, q_1) = 1 + (q_1 - 1)^2 b(J_z)$. In order to obtain the unitary representations for this q-deformed algebra we use the same condition of unitarity like in Sect.1, i.e. that the matrix elements of the operators $J_\pm J_\mp$ should be non-negative. Consequently, on the fundamental representation $|c, m>$, we obtain the corresponding conditions

$$< J_+ J_- > \;=\; \frac{1}{C_2}\Big(c - C_1(\tilde{g} - \tilde{g}^{-1})^2\Big) \geq 0, \tag{35}$$

$$< J_- J+ > \;=\; \frac{1}{C_2}\Big(c - C_1(\tilde{g} - \tilde{g}^{-1})^2 - \frac{2C_2}{h(q_1)}(\tilde{g}^2 - \tilde{g}^{-2})\Big) \geq 0, \tag{36}$$

where

$$C_1 = \frac{2^{4/3}}{(q_1 - q_1^{-1})^{4/3}(q_1^{1/2} - q_1^{-1/2})^{2/3}} \geq 0,$$
$$C_2 = \frac{2^{1/3}}{(q_1^{1/2} - q_1^{-1/2})^{1/3}} \geq 0,$$

and

$$\tilde{g} = q_1^{-1/4}\sqrt{f(J_z, q_1)} \geq 0.$$

Both conditions (35,36), reduce to second order inequalities concerning the function f and ask for the image of this function to be bounded in some regions, depending on c and q_1. For example eq.(35) is equivalent with the condition

$$f \;\in\; [L_1(c, q_1), L_2(c, q_1)], \tag{37}$$

with

$$L_{1,2}(c, q_1) = \sqrt{q_1}\Big(\beta(c, q_1) \mp \sqrt{\beta(c, q_1)^2 - 1}\Big),$$

with $\beta(c, q_1) = (1 + c/2C_1(q_1)$. These limiting values coincide for $c = 0$, $L_1(0, q_1) = L_2(0, q_1) = \sqrt{q_1}$ and tend to the limits $L_1 \to 0$, $L_2 \to \infty$, when $c \to \infty$, respectively. Consequently the condition (35) is fulfiled for c larger than a certain value, such that the image of f should be included in (L_1, L_2). The second condition, eq.(36), can also be written in the form

$$f \in [l_1(c, q_1), l_2(c, q_1)], \tag{38}$$

with

$$l_{1,2} = \sqrt{q_1}(\beta^2\alpha_+ \mp \sqrt{\beta^2\alpha_+^2 - \frac{\alpha_+}{\alpha_-}},$$

with $\alpha_\pm^{-1} = 1 \pm 2C_2/C_1(q_1)h(q_1)$. The analysis of eqs.(37,38) shows that for c larger than a certain positive value, $c_{min}(q_1)$, there are always nontrivial solutions for f, i.e. $l_2 > L_2 > l_1 > L_1$ for $c > c_{min}(q_1)$. We denote the boundaries of the allowed domain of f with $f \in (f_{min}(c, q_1), f_{max}(c, q_1))$. This interval exists for $c \geq c_{min}(q_1) > 0$ and its lenght increases with the increasing of c. This finaly proves that one can chose the functions f in eq.(34) such that their images are included in (f_{min}, f_{max}). Consequently, the unitarity conditions (35,36) are fulfiled for all the values of $f(J_z, q)$ in eq.(34), i.e. for all m eigenvalues and $q = q_1$. Since there are no limitations concerning $m \in Z$ or $\frac{1}{2}Z$ this proves the existence of infinite-dimensional unitary representations of $\mathcal{L}_{\tilde{g}}, q_1$ described by $m \in \mathbf{Z}$ or $\frac{1}{2}\mathbf{Z}$ (unbounded from below and above), $c \geq c_{min}(q_1)$, (continuous, bounded from below and unbounded from above) and g^2 given by eq.(34). In the following, we give a simple example of such a q-deformation. We can choose for the function f in eq.(34) the form

$$f(J_z, q_1) \quad = \quad \frac{f_{max}(q-1) - f_{min}(q_1)}{\cosh^2(J_z)} + f_{min}(q_1). \tag{39}$$

It is now easy to calculate the corresponding spectrum of $[2J_z]$, for this deformation. This spectrum is discrete, infinite dimensional, contains one accumulation point and it is bounded from above and below. Such spectra are very usefull in physical applications (atomic or molecular physics) because they can generalise atomic or molecular spectra where the distance between two consecutive lines continuously decreases. The Hidrogen atom has, as an example, such a kind of spectrum with its accumulation point at its superior bound. There is also interest for such spectra in conformal field theories.

We would like to do some comments at the end of Sections 2 and 3. We have shown that starting from the $su(2)$ Lie algebra symmetry, we could obtain deformed algebras (through q-deformation with $q = e^{is}$ in Sect.2 or through a general real q-deformation of the commutators in Sect.3), having infinite-dimensional unitary representations, some of them being irreducible too. The transition between the bounded unirreps $\mathcal{D}^j$ of $su(2)$ to the unitary representations of the corresponding deformation is made by the continuous variation of the parameter of deformation, in both cases. Consequently,

physical systems having dynamical symmetry close to that of $su(2)$ could be described in this formalism. We expect especially to obtain good results in the investigations of scattering processes, where the Hamiltonian could be expressed in terms of the deformed operators, like $[2J_z]$ or C. In this case, the parameter of deformation could play the role of a coupling parameter of the system, which allows the smooth transition from bounded to free states. A more complete picture could be obtained if one can construct differential representations of such deformed structures. In this way a Schrödinger picture can be obtained together with informations about the corresponding potentials and, the interactions. This is the task, for the above discussed case, in the following Section.

4 One-dimensional potential realisations for the complex deformation $su_{e^{is}}(2)$

In this Section we introduce a differential operator representation of $su_{e^{is}}(2)$, $s \in R$, acting on a two-dimensional space, in order to obtain a 1-dimensional Schrödinger equation and, consequently, equivalent potential picture for different values of s and m. We strictly follow the procedure given in [28], but we use a sort of "q-deformation" of the derivative with respect to one of the coordinates, as was early suggested by Biedenharn [4] and Macfarlane [5]. In this way one could use the $su_{e^{is}}(2)$ algebra as a dynamical symmetry close to some physical applications. We are interested especially in the classes of potentials which are connected with the eigenfunctions of the infinite-dimensional unitary representations obtained in Section 2.

We introduce a 2-dimensional coordinate space, parametrised with the polar coordinates $\phi \in [0, \pi)$, $r \in R$. The differential realisation is given by

$$J_{\pm} = e^{\pm i\phi}\left(\pm\partial_r + f_1(r)\left(\frac{1}{2}[2i\partial_\phi]\right) + f_2(r)\right), \tag{40}$$

$$J_z = -i\partial_\phi, \tag{41}$$

where the $\pm$ signs appear in order to satisfy the hermiticity conditions for $J_{\pm}, J_z$ and

f_1, f_2 are real smooth functions of r which have to be determined in order to re-obtain the commutators of $su_q(2)$. The operator $[2i\partial_\phi] = -[2J_z]$ is a sort of q-deformation of the derivative with respect to the ϕ coordinate. This q-deformation is defined by the formal action of the exponential operator (Taylor series) on exponential functions. By introducing eq.(41) in eq.(9) we have the action

$$[2i\partial_\phi]e^{im\phi} = -[2m]e^{im\phi}, \tag{42}$$

for $q = e^{is}$, s real, $[m] \in \mathbf{R}$. Consequently $[2i\partial_\phi]$ commutes with J_z and $[J_z]^\dagger = [J_z]$.

In the following, we deduce some properties of the operator $[2i\partial_\phi]$. We note that this operator is self-adjoint on the set of the functions of the form $|jm >= \Psi_{jm}(r,\phi) = R_{jm}(r)e^{im\phi}$ We introduce the auxiliar operators

$$\Omega_\pm = e^{\pm i\phi}[2i\partial_\phi]e^{\mp i\phi}, \tag{43}$$

which act through eq.(42) like

$$\Omega_\pm e^{im\phi} = -[2(m \mp 1)]e^{im\phi}. \tag{44}$$

We need some auxiliar relations, defined for any two generic quantities x_1, x_2, in the form

$$[x_1 - x_2] + (-1)^k[x_1 + x_2] = (-1)^k\eta^2[x_{3-k}]\Big[\frac{x_k}{2}\Big]^2 + 2(-1)^k[x_{3-k}], \tag{45}$$

and

$$[x_1 + x_2] + (-1)^k[x_1] = \eta^2\Big[\frac{x_2}{2}\Big]^k\Big[\frac{1}{3-k}\Big(x_1 + \frac{x_2}{2}\Big)\Big]^{3-k} + 2\Big[\frac{x_2}{2} + (k-1)x_1\Big], \tag{46}$$

with $k = 1, 2$ and $\eta = 2i \cdot sin(s)$. In order to realise the $su_{e^{is}}(2)$ algebra withthe operators $J_\pm, J_z$, given in eqs.(40-41), these operators must satisfy the commutators of this deformed algebra, i.e. eqs.(1,2). By using eqs.(45-48) it is easy to check that these operators satisfy eq.(1). For eq.(2) we have ($f_i' = df_i/d\rho$)

$$[J_+, J_-]e^{im\phi} = -[2m]\Big(f_1' + \frac{f_1^2}{2}\Big)e^{im\phi} + \mathbf{O}e^{im\phi}, \tag{47}$$

where the operator $\mathbf{O}$ acts on $e^{im\phi}$ in the form

$$\mathbf{O} = \Big[[2]f_1f_2\Big(1 + \frac{\eta^2[m]^2}{2}\Big) - \frac{[2]}{4}f_1^2\eta^2[2m][m]^2 + 2f_2' + \frac{f_1}{2}\eta^2[2m]\partial_\rho\Big]. \tag{48}$$

Consequently, by identifying eq.(2) with eq.(47), we obtain two conditions

$$f_1' = -1 - f_1^2 cos(s), \tag{49}$$

$$\mathbf{O} R_{cm} e^{im\phi} = 0. \tag{50}$$

The action of the Casimir operator of $su_{e^{is}}(2)$, eq.(3), on the basis $R_{cm}(r)e^{im\phi}$, reduces to

$$\Big(-\partial_r^2 + \frac{f_1}{2}\Big(2 + \eta^2\Big[m - \frac{1}{2}\Big]^2\Big)\partial_r + [2m][2(m-1)]\frac{f_1^2}{4} -$$

$$\frac{f_1 f_2}{2}(2+\eta^2)[2m-1] - \frac{f_1'}{2}[2m] + f_2^2 + f_2' + [m]^2 +$$

$$\Big[m - \frac{1}{2}\Big]\Big) R_{cm}(r) = c R_{cm}(r), \tag{51}$$

for any m, in the corresponding representation labeled by c.

The first condition, eq.(49), is similar with the corresponding condition usualy used in differential realisations of $su(2)$ or $su(1,1)$ in AST [28,40], excepting the deformed coefficients like $[2]/2$, instead of 1. This equation has unique solutions, independent of the quantum numbers. There exist three different cases, accordingly to the range of s. For $cos(s) < 0$ (which includes the limiting case $s \to \pi$ of $su(1,1)$) we have

$$f_1(r) = \pm\frac{1}{\sqrt{-cos(s)}} \cdot \begin{cases} tanh(\mp\sqrt{-cos(s)}r + d) \\ 1 \end{cases}, \tag{52}$$

where d is a constant of integration. For $cos(s) > 0$ (which contains the limiting case $s \to 0$ of $su(2)$) we have only one solution

$$f_1(r) = \pm\frac{1}{\sqrt{cos(s)}} tg(\mp\sqrt{cos(s)}r + d), \tag{53}$$

and no constant real solution. Finally, for $cos(s) = 0$ we have

$$f_1 = -r + d. \tag{54}$$

In order to have the continuity of f_1 between the above solutions, with respect to $s \in [0, \pi]$, we need to fix $d = 0$. We note that the solutions in eqs.(52-54) are similar with those obtained in the undeformed case, excepting the scalling term $\sqrt{\pm cos(s)}$.

The second condition, eq.(50), is more special due to the fact that the operator **O** from eq.(48) contains the derivative with respect to r. This a special effect due to the deformation (does not occure in the undeformed case), i.e. due to the terms containing $\eta \neq 0$. This special dependence introduce a coupling between the function f_2 and the eigenstates R_{cm}. Consequently, we can see that, due to the deformation, a different situation compared with the undeformed case appears: one has the possibility of obtaining other different potentials from the $su_{e^{is}}(2)$ dynamical symmetry, in comparison with the undeformed case. On the other hand it is more difficult to realise the eigenproblem for the Casimir operator into a Schrödinger equation, due to the fact that there are two coupled differential equations for the eigenstates, i.e. eqs.(50,51).

Limiting cases

We analyse this differential realisation for three (simpler) limiting cases, i.e. when eqs.(50) and (51) become decoupled due to the canceling of the last term in the RHS of eq.(48). This happens for $s \simeq \pi$ (any m), $s \simeq \pi/2$ ($m \in Z$) and $s \simeq 0$ (any m) with $s \in (0, \pi)$. We supose that for all these situations the coefficients η^2 and/or $[2m]$ are enough small to be neglected. Consequently, in these approximations, eq.(50) can be integrated independent of R_{cm} and we obtain an approximate differential equation for f_2

$$\frac{[2]}{2} f_1 f_2 \quad = \quad -f_2^{'}. \tag{55}$$

By integrating this equation with f_1 given by eqs.(52) we have, for the first limiting case, ($s \simeq 0$)

$$f_2(r) \quad = \quad \begin{cases} \frac{F_1}{cosh(\mp\sqrt{-cos(s)}r+d1)} \\ F_2 e^{\pm\sqrt{-cos(s)}r} \end{cases} . \tag{56}$$

For $s \simeq 0$ limiting case ($cos(s) > 0$) we have

$$f_2(r) \quad = \quad F_3(cos(\mp\sqrt{cos(s)}r + d2)). \tag{57}$$

Finally, in the case $s \simeq \pi/2$ we have

$$f_2(r) = F_4 = const.$$

The constants of integration F_{1-4} and $d1, d2$ can be choosen such that f_2 should be continuous with respect to s. With the above solutions for the functions f_1 and f_2 we can construct the differential form of the Casimir operator, close to the limiting cases $s \simeq 0$ and $s \simeq \pi$ (i.e. in the approximation $\eta^2 << 1$)

$$Ce^{im\phi} = \Big(-\partial_r^2 - f_1\partial_r + [2m][2m-2]\frac{f_1^2}{4} - f_1 f_2[2m-1]$$

$$- \frac{f_1^{'}}{2}[2m] + f_2^2 + f_2^{'} + [m]^2 + \Big[m - \frac{1}{2}\Big]^2\Big)e^{im\phi}, \quad (58)$$

and close to the intermediate limiting case $s \simeq \pi/2$, $m \in Z$

$$Ce^{im\phi} = -\partial_r^2 - \frac{f_1}{2}(2 + (-1)^{m+1})\partial_r + (-1)^m \frac{f_1 f_2}{2}(2 + \eta^2) +$$

$$f_2^2 \quad + \quad f_2^{'} + [m]^2 + 1)e^{im\phi}. \quad (59)$$

Both eqs.(58,59) can be reduced to a 1-dimensional Schrödinger-like equation

$$(-\partial_r^2 + V(r; m, s))\Psi_{cm}(r) \quad = \quad c\Psi_{cm}(r), \quad (60)$$

$\Psi_{cm}(r, \phi) = R_{cm}(r)e^{im\phi}$ by using the substitution: $R_{cm}(r) \rightarrow a(r)R_{cm}(r)$. For example, in the case of eq.(58) we choose

$$a(r) = a_0 exp\Big(-\int f_1(r)dr\Big), \quad (61)$$

and we obtain a potential dependent of r and of the parameters m and s

$$V(r; m, s) = -\frac{a^{''}}{a} + [2m][2m-2]\frac{f_1^2}{4} - f_1 f_2[2m-1] -$$

$$\frac{f_1^{''}}{2}[2m] \quad + \quad f_2^2 + f_2^{'} + [m]^2 + \Big[m - \frac{1}{2}\Big]^2, \quad (62)$$

with f_1, f_2 given by eqs.(52-54) and (56,57), respectively. Eq.(62), gives a posible equivalent 1-dimensional potential picture (Schrödinger) for the limiting cases $s \simeq 0$ and $s \simeq \pi$ and close to them. In the case $s \simeq \pi/2$, $m \in Z$, one can use the same type of substitution and $a(r)$ differs only by a constant factor in the exponential, in eq.(61).

For exemplifying we present in Figs.4-6 some potential shapes drawn for few different limiting values of s.

In Fig.4a the potential $V(r;m,s)$ given by eq.(62) is presented in the case $cos(s) > 0$, for different values of $s \in (0, pi/2)$, $s \simeq 0$, $s \simeq \pi/2$ and $m = 1$. The potential consists in an infinite series of periodic copies of well potentials valleys of infinite depth. The aspect of V does not modifies essentially when s modifies from zero to $\pi/2$. In Fig.4b the same potential is presented for fixed $s = 0.25$ and different $m = 1, 3/2, 2, 5/2, 3$ and $7/2$. We note that the modification of m, below or above a limiting value (situated next to $m = 2$), drastically transforms the aspect of V: from a countable succesion of infinite-depth potential wells (at small m's) into a similar sequence of positive poles. The shapes for $m = 1, 3/2$ and 2 are similar with the potential of a quantum particle (electron) in an atomic crystal. One knows that in this case the spectrum is spread out in bands of closely spaced levels for c (in between the wells) and continuous for c above their common tops. In the present case the bands are generated by the discrete representations of class $\underline{(1)}$ and $\underline{(2)}$. The limiting value for the transition bands/continuous increases with s, as predicts the analysis of these representations, Fig.1a. For $m = 2, 5/2, 3$ and 7 we obtain a sum of positive potentials which are periodicaly distributed on the r axis. This results from the fact that in the expression, eq.(62) of V there appear only the functions $f_{1,2}(cos(s)r)$, given by eqs.(53,57). All these functions have the same trigonometric structure, with the same period. This is a consequence of the fact that, due to the decoupling of f_2 and R_{cm}, f_2 can be directly obtained from eq.(55) and it depends only on f_1. Thus, the resulting potential is only a sum of potentials with the same space-period. Consequently, the spectrum is absolut continuous [41], the eigenvalues are of Bloch-type states in which the paticle is found with infinitesimal probability in every finite region of the axis. The q-deformation, in this case, (s close to 0 or to $\pi/2$) does not bring but quantitative modifications against the undeformed case. Here s can be regarded as a fitt parameter, only.

In Fig.5 we present some potential shapes in the range $cos(s) < 0$, close to the limits $s \simeq \pi/2$ and $s \simeq \pi$, constructed by introducing the first solutions of $f_{1,2}$ from eqs.(52,56) in eq.(62). The resulting potentials are no more periodic and they have exponential (hyperbolic) behaviour. In Fig.5a we show the potential for a fixed $m = 1$ and for different s. In this figure the potential shapes with $s = 3.05$ and 3 ($\simeq \pi$) are drawn translated along the vertical axis, in order to have a better pictorial view. We note that in the both limits, the potentials modify drastically: closed to $s = 3$, i.e. s

close to an integer, behave as pure repulsive for $r > 0$ and, close to $s \simeq \pi$ the potential always has a pocket, close to $r \simeq 0$. The first examples are similar with Pöschl-Teller-like potentials and the latter cases give shapes similar with a negative signed harmonic oscillator potential.

In Fig.5b the same potentials as in Fig.5a are presented, for fixed $s = 3$ ($\simeq \pi$) and different m's. For example, for $m = 1$ the potential is practicaly symmetric in r, Pöschl-Teller-like. For larger values of m the potential become rather antisymmetric in r and consists in a bounded valley (close to $r = -1$ for $m = 3$ and close to $r = 0.5$ for $m = 5/2$) followed by a shoulder. In the $s \to \pi$ limit these potentials coincide with a Pöschl-Teller one, like in the undeformed case.

By introducing the second solutions of $f_{1,2}$ from eqs.(52,56) in eq.(62), we get now Morse-like deformed potentials. In Fig.6a we present such shapes for $s \simeq \pi/2$ and fixed $m = 1$. Here, all the potentials are also drawn translated along the vertical axis such that they have the same asymptotic value for $r \to -\infty$, i.e. $V(s = 3.04) \to V(s = 3.04) - 94.8$, etc. For values of the Casimir eigenvalues samller than the asymptotic limit, the spectrum is discrete ($c_2 < c < c_0$), for $c > c_0$ the spectrum is continuous and for c bellow c_2 there exist no states. This inferior limit c_2 decreases with the increasing of s, in the range $s \in (\pi/2, \pi)$, as prescribed in Sect.2 for type $\underline{(2)}$ and $\underline{(2)}$ representations.

In Fig.6b we present the same potential for fixed $s = 3.05$ ($\simeq \pi$)and different values for m. We also remark here a very strong shape-dependence of the potential with m. For $m > 2$ the potential is Morse-like and the depth of the valley increases with m like in the case of the undeformed limit [28]. For $m < 2$ there are no more valleys and the potential is purely atractive. In this case the spectrum is absolute continuous. This is again in agreement with the analysis made in Sect.2.

For the intermediate case $s = \pi/2$ we have $[2] = 0$ and from eq.(54) and eq.(62) we obtain an exact harmonic oscillator potential. If, in addition, $m \in Z$ and $F_4 = 0$ we obtain $V = const$. This is also in perfect agreement with the consequences of the analysis of the representations for this limiting case $s \to \pi/2$ in Sect.2, ($e(2)$ dynamical symmetry breaking). This situation describes a 1-dimensional free particle and can be used similar with the Euclidean connection [28], as the asymptotic limit for a scattering process.

By taking into account all these examples we can conclude that, the parameters s and m provide the $su_{e^{is}}(2)$ model with a strong symmetry breaking effect: the corresponding Schrödinger picture evoluates from periodic potential (Bloch-like) made of negative singularities into a periodic series of positive poles, or into different bounded potentials allowing localised states (q-deformed Pöschl-Teller-like, Morse-like, harmonic oscillator like, etc.), or, finaly, into constant potential (free asymptotic states). This variety of the obtained shapes, when one modifies continuous s, and discretely m, gives a wide area of possible 1-dimensional Schrödinger pictures, all unified into the same dynamical symmetry, $su_{e^{is}}(2)$. We note that similar results were obtained in the references [23,24], concerning the deformation of an harmonic oscillator into a minus Pöschl-Teller-like potential (similar with one of the cases presented in Fig.5b).

<u>General solutions</u>

The general solutions of the system of eqs.(50,51) bring the most general class of equivalent exact solvable potentials obtained through the realisation (40,41). However, we mentione that there exist many other possibilities of differential realisation of $su_q(2)$ (starting with [5] up to a large variety of such realisations in the present literature). In the above Subsection we have presented some limiting cases, when, due to the decoupling of the two involved differential equations for the function f_2 and for R_{cm}, four classes of potentials, are obtained. These potentials are only combinations of trigonometric ($cos(s) > 0$), exponential ($s = \pi/2$) and hyperbolic ($cos(s) < 0$) functions.

The general case, when the coefficient $f_1\eta^2[2m]$ in eq.(48) is no more neglected, is much more interesting. Due to the coupling of the differential equations for f_2 and the radial wavefunction we need a special method of solving these problems. First, f_1 is always given by eq.(49). From eqs(48,50) we have

$$R^{'}_{cm} = A(r)R_{cm},$$

$$R^{''}_{cm} = A^{'}R_{cm} + AR^{'}_{cm}, \tag{63}$$

with

$$A = \frac{[2]}{2}[m]^2 f_1 - [2]\left(\frac{2}{\eta^2[2m]} + \frac{[m]^2}{[2m]}\right) f_2 - \frac{4}{\eta^2[2m]}\frac{f_2^{'}}{f_1}. \tag{64}$$

By introducing R''_{cm} and R'_{cm} from eq.(64) in eq.(51), $C(f_1, f_2, m, s)R_{cm} = cR_{cm}$, we obtain a non-homogenous, non-linear differential equation with variable coefficients for f_2, in the form

$$\begin{aligned} a_m(r)f_2'' + b_m(r)f_2' + c_m(r)(f_2')^2 + d_m(r)f_2'f_2 + \\ e_m(r)f_2^2 + h_m(r)f_2 + g_m(r) &= c \end{aligned} \tag{65}$$

where the functions $a_m(r)...g_m(r)$ are obtained as combination of the terms of A, A^2, A' and of those terms in the LHS of eq.(51) which do not contain the derivatives ∂_r. The formal solution $f_2(r; m, s, c)$ of eq.(65), together with $f_1(r; m, s)$ from eq.(49), introduced in eq.(58) and with the help of the substitution (61) give a corresponding potential $V(r; m, s, c)$, and the eigenfunctions

$$R_{cm} = R_0(c, m, s)exp\Big(\int\Big(\frac{[2]}{2}\eta^2[2m][m]^2 f_1 - [2]f_2(2 + \eta^2[m]^2) - 4\frac{f_2'}{f_1}\Big)dr\Big). \tag{66}$$

The next step is to generate all the other states of the corresponding unitary representations from which the first obtained solution, eq.(66), belong, by acting with $J_\pm$ on it. At present we do not know if this method allows the obtaining of all the eigenstates, but we can check its consistency at the algebra level. Suppose we have calculated the solution f_2 of eq.(65) and consequently we have obtained a first state given by eq.(66). We obtain a new state in the form

$$\tilde{R}_{cm}e^{i(m+1)} = J_{+cm}R_{cm}e^{im\phi}, \tag{67}$$

where we denote with $J_{\pm cm}$, C_{cm} the differential operators eq.(40,41) in which we have introduced the solutions $f_{1,2}(r; m, s, c)$. We have

$$C_{cm}\tilde{R}_{cm}e^{i(m+1)\phi} = c\tilde{R}_{cm}e^{i(m+1)\phi}, \tag{68}$$

because $[C_{cm}, J_{\pm cm}] = 0$, and from eq.(2)

$$[J_+, J_-]_{cm}\tilde{R}_{cm}e^{i(m+1)\phi} = [2J_z]J_{+cm}R_{cm}e^{im\phi}. \tag{69}$$

The eqs.(1,2) can be written in an equivalent form by using the example 1.5.3 given by Majid in [32]

$$q^{\pm 2J_z}J_\pm q^{\mp 2J_z} = q^{\pm 2}J_\pm.$$

We obtain from eqs.(69,70)

$$[2J_z]J_{+cm}R_{cm}e^{im\phi} \;=\; [2(m+1)]\tilde{R}_{cm}e^{i(m+1)\phi}. \qquad (70)$$

From eqs.(69,71) we have then

$$J_{+cm}R_{cm}e^{im\phi} = R_{c,m+1}e^{i(m+1)\phi},$$

which is in agreement with the action of J_+ in the representation (c, m). The same thing happens with J_-. In coordinates this reads

$$\Psi_{c,m\pm1}(r) \;=\; \left[\pm R'_{cm}(r) + \left(-\frac{f_1(r)}{2}[2m] + f_2\right)R_{cm}(r)\right]e^{i(m\pm1)\phi}. \qquad (71)$$

In conclusion of this section, the present method can generate different exact 1-dimensional potentials and their corresponding eigenfunctions. The problem of obtaining the full spectrum is strongly dependent on the value of s, i.e. when q is or is not a root of 1. A complete investigation of such realisations is in course of finalisation and it will be the subject of a following paper.

Remarks

1) One interesting new point introduced here in the differential realisation of $su_{e^{is}}(2)$ is the resulted coupling between f_2 and R_{cm} in eq.(48). In the undeformed case ($\eta = 0$), $f_{1,2}$ can be determined uniquely, and independent of R_{cm}. Consequently one has a given $V(f_{1,2}(r), m)$ and can solve the eigenproblem for this potential, i.e. to find the eigenstates according to the unirreps of the undeformed algebra. More, since eq.(50), for $s = 0$, relates f_1 and f_2, if they are periodical functions, they should have the same period. This case gives only periodic potentials with Bloch structure, continuous spectrum (bands or full continuous) and delocalized eigenstates. The $q = e^{is}$ deformation allows, through the above mentioned coupling, the occuring of new situations. Cases similar with the undeformed case are encountered for $s = \pi\frac{p}{l}$ (p, l integers, q a root of 1), when, for certain sets of m the coupling term $\eta^2[2m]$ vanishes. For $s \neq \pi\frac{p}{l}$ (q not a root of 1), one can choose an arbitrary function for f_2 and can solve R_{cm} from eq.(50) only. Then, eq.(51) is automatically fulfilled (since $R_{cm}e^{im\phi}$ belongs to the space of the representations of $su_{e^{is}}(2)$ and c results. In such situations

we can obtain, for a general potential $V(f_{1,2}(r), m, s)$ only some of the solutions of the corresponding Schrödinger equation.

If $s = \pi \frac{p}{l}$ we obtain different solutions R_{cm_k} for different m_k such $[m_k] = [m_{k'}]$, $[2m_k] = [2m_{k'}]$, ... etc. If $s \neq \pi \frac{p}{l}$ one can directly obtain only one solution for each potential. The rest of the solutions for the same potentials are obtained by the action of the generators $J_\pm$, as shown above.

We give an example: suppose f_1 is a periodical (of period $\sqrt{cos(s)}$) and we look for the possibility of obtaining localised solutions R_{cm}. Being localised, these solutions contain almost all the Fourier components in the variable r and, consequently, by solving eq.(48) with respect to f_2, for such R_{cm}, we can obtain periodical solutions for $f_2(r)$ but in an incommensurate period with respect to $\sqrt{cos(s)}$. As a consequence, the potential will be a random potential and we have simulated, in this way, the Anderson localization process [41]. On the contrary, if R_{cm} is delocalized and consequently a periodic function, by following the same methods f_2 will be also a periodic function and $f_{1,2}$, V and R_{cm} functions will have the same period, or integer multiples or sub-multiples of the original one. In this case we have obtained the case of commensurate potential, as one expects from the theory of solvable potentials. These two opposite aspects are controled through the s parameter. For example, when q is a root of 1 (special cyclic representations from the point of view of q-groups [29-32]) there exist values for m such that $[2m] = 0$. In this case the condition eqs.(48,50) ask for $f_1 f_2 = 0$. We can find a possible solution for this last condition, in the form

$$f_1(r) = \sum_{k \in I_1} C_k^1 \Phi_k(r),$$

$$f_2(r) = \sum_{k \in I_2} C_k^2 \Phi_k(r), \tag{72}$$

where $I_1 \cap I_2 = \{0\}$, $C_k^{1,2}$ are constants and $\Phi_k(r) = 1$ for $k + \epsilon < r < (k+1) - \epsilon$, k integer, $0 < \epsilon < 1$, and zero in the rest (wavelet functions). In this case f_1 and f_2 have disjoint support and the $f_1 f_2 = 0$ condition is fulfilled. For $\epsilon \to 1$, this example can provide exactly the situation of periodic but incommensurable potentials [41].

2) We want to note other two observations. First that the transition from $su(2)$-like potentials (finite depth) towards $su(1,1)$-like potentials (infinite wells) is performed

through $su_{e^{is}}(2)$ in a continuous way, by s. In this modality the q-deformed algebraic structure is a dynamical symmetry for a larger class of potentials, including both limiting cases (bounded and scattering states). Another interesting point arises when looking at the turning point between these two different situations (change of sign of $cos(s)$). From the potential point of view this separations in the two regions of deformation clasifies the potentials in "trigonometric"-like and "hyperbolic"-like potentials, due to the solutions of the functions f_1, f_2 arrising in the differential realisation of the generators. This condition is independent of the values of the quantum numbers c and m. By regarding at eq.(21) we remark that the same factor $cos(s)$ occurs in the Casimir expression, i.e.: $\frac{\cos s}{\sin^2 s}$, which is the multiplicative factor in front of $\sin 2(sJ_z)$. So, s can transform the Casimir element from a positive defined one into an indefinite one. For the case $\cos(s) < 0$ (which contains the $su(1,1)$ limit), we have, on one hand, hyperbolic-like potentials (e.g. Pöschl-Teller or deformation of it) which allow both discrete and continuous spectra, infinite-dimensional. On the other hand, in the same range for s, due to the fact that $c > \frac{1}{\sin^2 s}$, for any described series in Section 2, the "surfaces" described in the end of Sect.2, defined by the condition $c = constant$, are open surfaces (eq.(21) similar with the hyperboloidal surfaces connected with SU(1,1). The reverse is also true: for $cos(s) > 0$ (which contains the $su(2)$ limit) the resulting potentials are of trigonometric type (depend on trigonometric functions) and have poles which divide the axis r in infinite wells potential. In this latter case the corresponding "topology" (Sect.2) is described by an infinite reunion of compact spheres, similar with the $SU(2)$ topology. This unexpected similarity between: the structure of the unitary representations, the "shapes" associated with the Casimir operator eq.(19), and the behaviour of the corresponding potentials, addres us the information and believe that such approach of complex q-deformation of $su(2)$ is really a candidate for the unification of the bounded and scattering systems in the same dynamical symmetry.

5 Conclusions

In the present paper we have investigated some problems connected with the complex deformation of the Lie algebra $su(2)$ with $q = e^{is}$ and through a general real deformation of $su(2)$, as a Hopf algebra, too.

Using the usual trigonometric deformation [1-9,11-16,21-23] for the q-algebra $su_q(2)$, we have constructed and clasified some of the coresponding unitary representations and we have shown the existence of the possibility of an extension of the bound representations of $su(2)$ into three classes of representations: continuous infinite-dimensional and discrete ones. We note that the present $q = e^{is}$ deformations alow the "transition" of some of the properties of the Lie algebra $su(2)$ (unirreps, Casimir operator) towards those of the Lie algebras $su(1,1)$ and $e(2)$. A pictorial illustration of the surfaces of constant value of the q-deformed Casimir operator eigenvalues depending on s and c is described in a "formal" vector space of the generators of $su(2)$. We note that even this example underline the similarity between $su_{e^{is}}(2)$ and $su(1,1)$, for certain values of s. Consequently there are chances for the application of such smooth deformations in the AST theory, esspecially when one needs to connect the bound and the scattering states in the same unifying (q-deformed dynamical symmetry) picture.

We have proved the existence of an exact analytical solution for a real generalised deformation functional $[2J_z]$, which allows the deformation of the commutator relations of $su(2)$ together with the introduction of a Hopf algebra structure. This result (infinite dimensional spectra) shows that even for real deformations one can modify the compact caracter of $su(2)$.

We have obtained a realisation of $su_q(2)$ in terms of q-differential operators which could be related to an exact Schrödinger equation and a corresponding equivalent potential picture. We have solved the corresponding Schrödinger equation in some limiting cases and we have shown some examples of deformed potentials. For some particular values of s we have found different shapes of potentials of physical interest (solid state, nuclear, atomic and molecular interactions) like the examples given in Figs.4-6. One can see from these examples that one can transform, through the smooth variation of s, an infinite well potential into a finite one, like was first suggested in [24].

Consequently, due to the present approach, new potential shapes could get their algebraic analogue in the corresponding nonlinear q-deformation. The q-deformation of the bound states towards scattering states can be applied to processes like scattering or decays in which the physical system completely changes its symmetry. The deformed Pöschl-Teller potential could describe electrons in the valence band and the equivalent deformed potential the electrons in the corresponding conduction band. As an example one can use these new potentials also in the physics of semiconductor junctions. For example, the fact that, due to the coupling in eq.(48), the potential depends on all the quantum numbers, $V(r; c, m, s)$, one could describe the dynamics of the "polaron" in solid state structures or the deuteron states with one bound level (self-localised states). Direct applications in the theory of exact solvable 1-dimensional potentials with random or incommensurate structure can also be found. Since we have found some similarities between the rational/irrational form of the deformation parameter s/π with the commensurate/incommensurate character of the resulting potentials, we consider that exists a possibility to connect the s parameter with the order parameter of such potential models (which describes the random or ordered statusof a lattice model. In fact from eq.(40), regarded as an eigenvalue problem, we can note a direct connection with driven-rotator or other type of random oscillators, described by the eq.(2) in reference [41]. In this sens it will be interesting to further investigate the unitary representations of $su_{e^{is}}(2)$ when s/π is not rational (q not a root of 1) but belongs to a Liouville irrational class [41]. On the other hand, in the case when q is a root of 1 and the representations have special distinguished properties and implication for the product of the representations, we expect, via the above considerations, interesting posibilities of investigations of random/incommensurate potentials for many-body problems. And further applications are however, needed.

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Figure captions

Fig.1 Figures 1a-1b present pictorial views of some examples of unitary representations of $su_{e^{is}}(2)$ for $s = 1.013 < \pi/2$. The horizontal axis contains the values of m (integers and half-integers) and on the vertical axis contains the values of c, the eigenvalue of the Casimir operator. We plot with horizontal lines the limiting values for c, which separate the three main classes of unirreps, i.e.: $c_0 = \frac{1}{\sin^2 s}, c_1 = \frac{1}{4\sin^2(s/2)} c_0 = \frac{1}{\cos^2(s/2)}$. In the figures these classes of unirreps are labeled by the numbers (1)-(3) in the right part of each plot, and the arrows indicate the ranges for c. The lower bounds of c, depending on m, (RHS of eq.(12)), are ploted with continuous lines denoted with **1** and **2** for the sign $\pm$, correspondingly. functions.

(a) Here we present the infinite dimensional unirreps (class (1)), having their lower bound at $c = c_0$. The 14 full dots in the area $c > c_0$ are an example of such an unirrep for an arbitrary c. Their range is marked with an arrow (1) in the right side of the figure. In the same figure we plot the mixed series, (2) in the range $c_1 < c < c_0$ (see the arrow (2) in the right side of the figure). The full horizontal segments, having arrows at their ends, represent the sequences of the allowed/forbiden intervals for m, i.e. $J_\delta, J_f, J_\Delta, ...$, for a certain value of c, in the range $c \in (c_0, c_1)$. We represent with full dots an example of an infinite-dimensional unirrep belonging to class 2a, for $k = 1$, and with stars a finite dimensional unirrep for $N = 4$, class 2b. Though the space between the values of m is always 1, in order to present these two situations in the same picture, we have used two different values for s (for the dots and the stars)but we have ploted them together, within the same scale.

(a) We present here the range $c \in (c_1, c_2)$ for the discrete unirreps, class (3) (see the arrow in the right side of the figure). Again the full line with arrows shows the allowed/forbiden intervals and the dots represent an example of such a finit-dimensional unirrep for a certain c in the range, and for $N = 6$.

Fig.2 In this figure we present the dependence of the eigenvalues of $[J_z]$, $[2m]$, against the deformation parameter s. One can see that starting from the undeformed

case ($s = 0$), after many oscillations, in the final limit $s = \pi$, we obtain exactly the changing of the sign, for the values of m, i.e. even values in this illustration, which exemplifies the similarity with the unirreps of $su(1,1)$. We note that for some special values of s, like $\pi/k = \pi/2, \pi/3, \pi/4$ etc, the level cross and we have a pictorial view of what was calculated in Section 2, for the series 2a, case when there are exactely k distinct values for all $[2m]$.

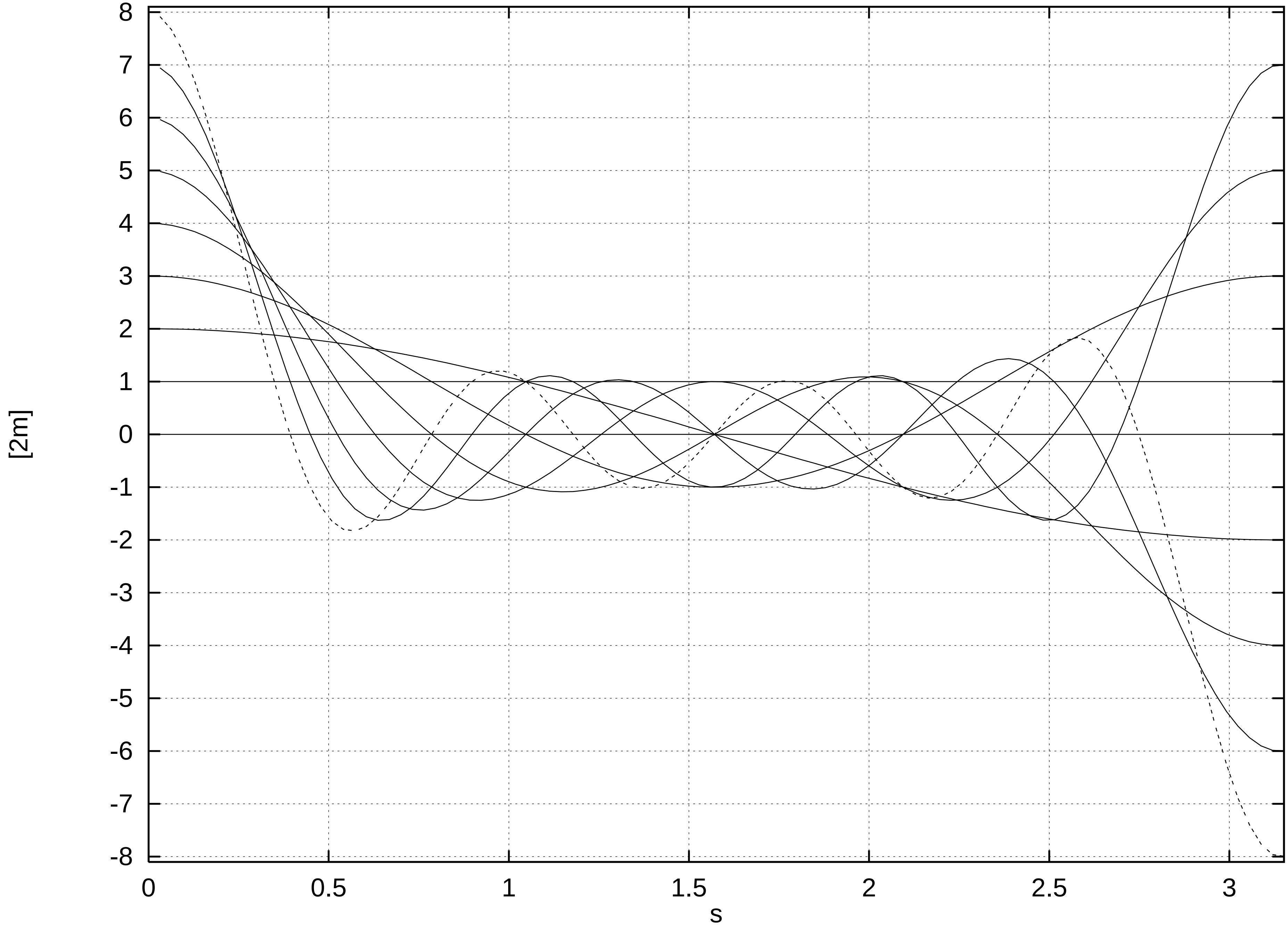
Fig.3 In this figure we present a cross section ($J_y = 0$) in the formal vector space (J_x, J_y, J_z), for the surfaces of constant value of the Casimir operator (J_z-horizonat axis, J_x-vertical axis). This illustrates graphicaly the behaviour of the Casimir eigen values for different valus of s. The three disjoint bigger circles represent the disconected surfaces which appear in the case $\cos s > 0$. The empty spaces between these spheres result from the fact that the square root which represent the expression of J_x is complex. The central smaller sphere represents the limiting case of $s = 0$, i.e. the spherical surface of the SU(2) manifold. For larger values of s, these spheres become "closer" one to each other and when $\cos s < 0$ they join into one unique surface, (the two pairs of continuous ondulated open lines in figure, computed for two different values of $s \in (\pi/2, \pi)$), homotopical with the hyperboloidal shape associated with the SU(1,1) manyfold. This $SU(1,1)$ hyperboloidal geometry is presented here by the two extreme curves, each having one single minimum (maximum) respectively.

Fig.4a and 4b Potential shapes, ploted against r in the case $cos(s) > 0$, $m = 1$. having periodical structure. In Fig.4a one can see the difference between the limits $\simeq 0$ and $s \simeq \pi/2$. In Fig.4b we ploted for fixed $s = 0.25$ (rational, i.e. q a root of 1) and different m's. One can see the transition between negative and positive poles of the potential, through the variation of m.

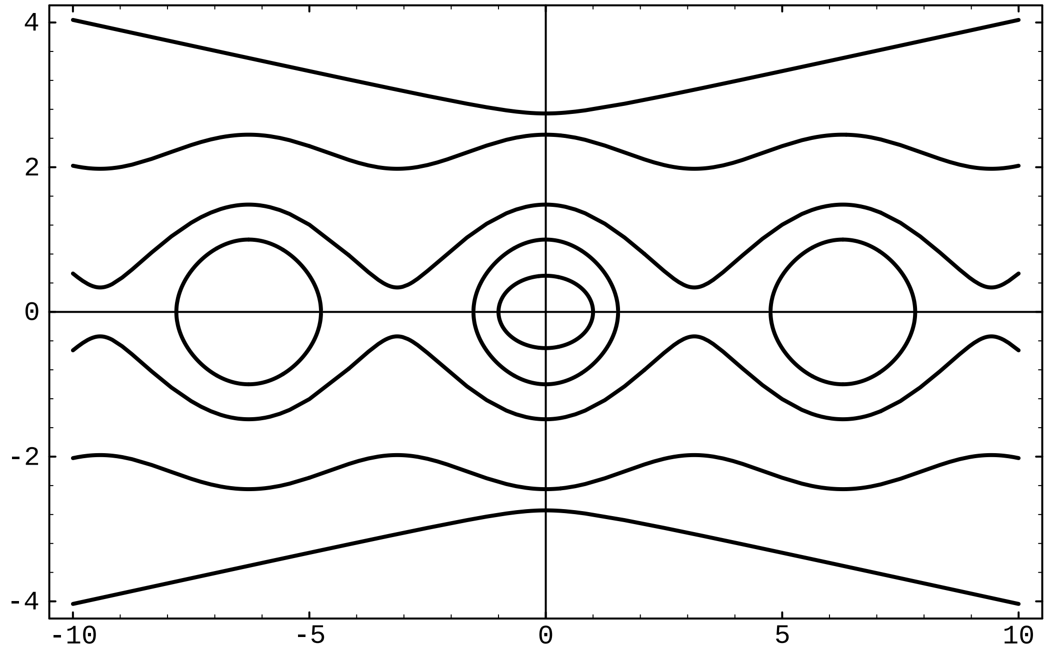
Fig.5a and 5b Potential shapes ploted against r in the case $cos(s) < 0$, for the first set of solutions of eqs.(52,56). In Fig.5a we have fixed m and variable s in the range $\pi/2, \pi$. One can see the transition from Pöschl-Teller-like potentials (s far from the limits) to (negative signed) harmonic oscillator-like potentials ($s \simeq \pi/2$, $s \simeq \pi$) In Fig.5b for fixed $s = 3$ and different values for m, the dependence of the shape with m

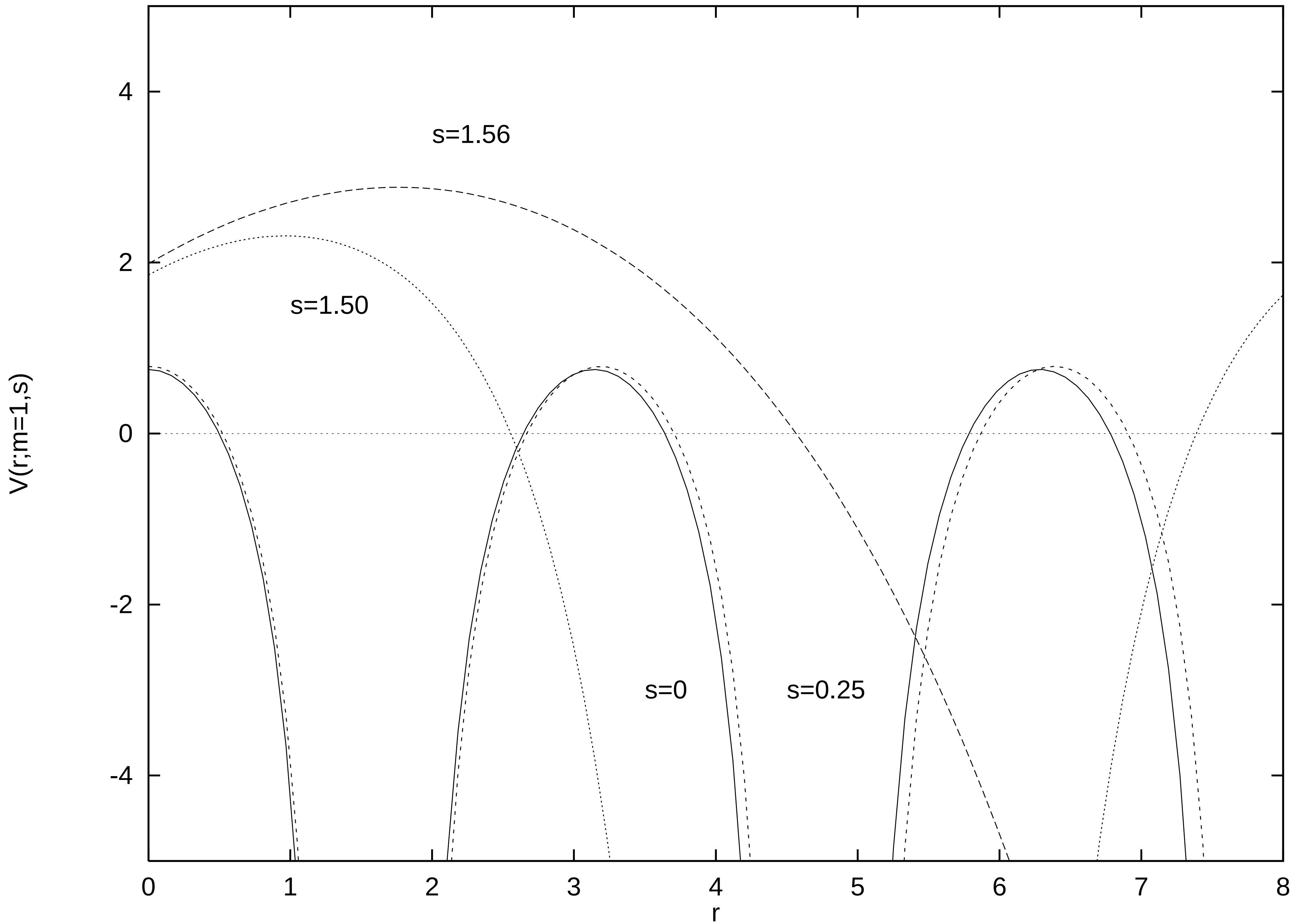
is presented. The difference between integer and half-integer m results in the symmetry of the shapes against $r = 0$.

Fig.6a and 6b Potential shapes ploted against r, also in the case $s \in (\pi/2, \pi)$, but for the second pair of solutions in eqs.(52,56). In Fig 6a, for fixed $m = 1$ and variable $s \in (\pi/2, \pi)$ the potentials are similar with Morse potential. In Fig.6b for fixed $s = 3.05$ (closed to π) and different m values. the potentials transfor from Morse-like ($m > 2$) into harmonic oscillator-like ($m < 2$).

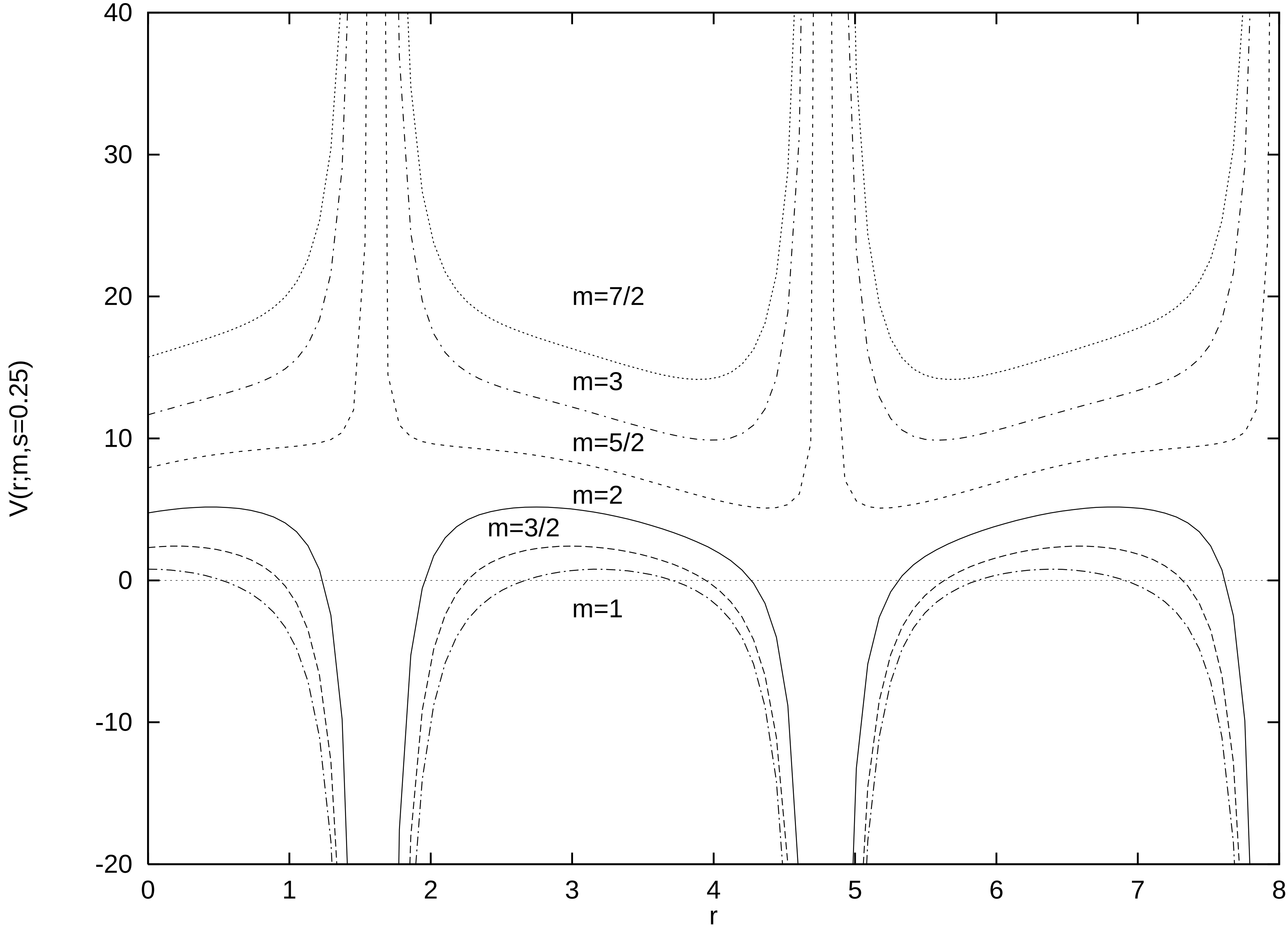


8
7
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
-6
-7
-8
[2m]
0
0.5
1
1.5
2
2.5
3
s

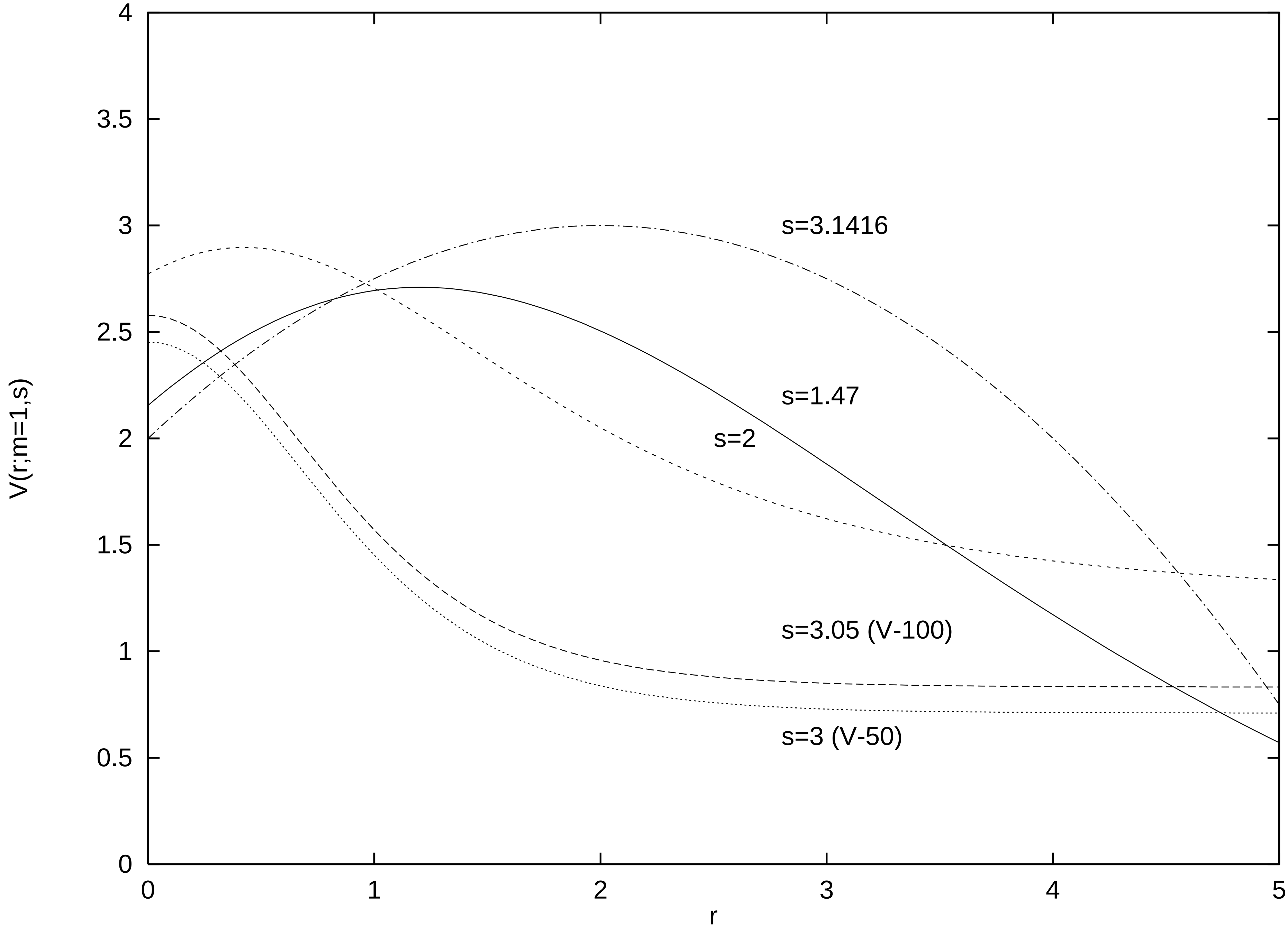

4
2
0
-2
-4
-10
-5
0
5
10



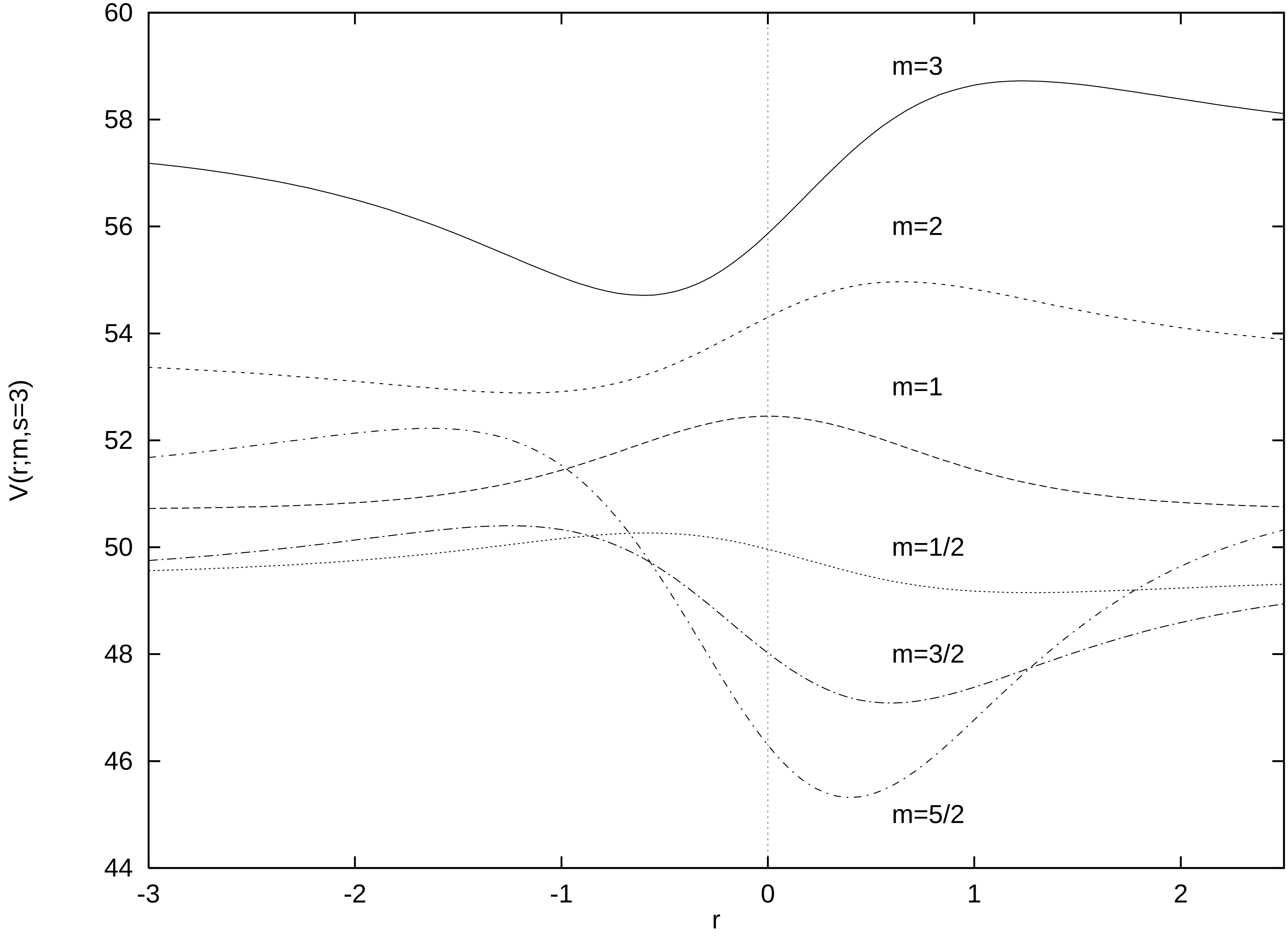
4
2
0
-2
-4
V(r;m=1,s)
s=1.56
s=1.50
s=0
s=0.25
0
1
2
3
4
5
6
7
8
r



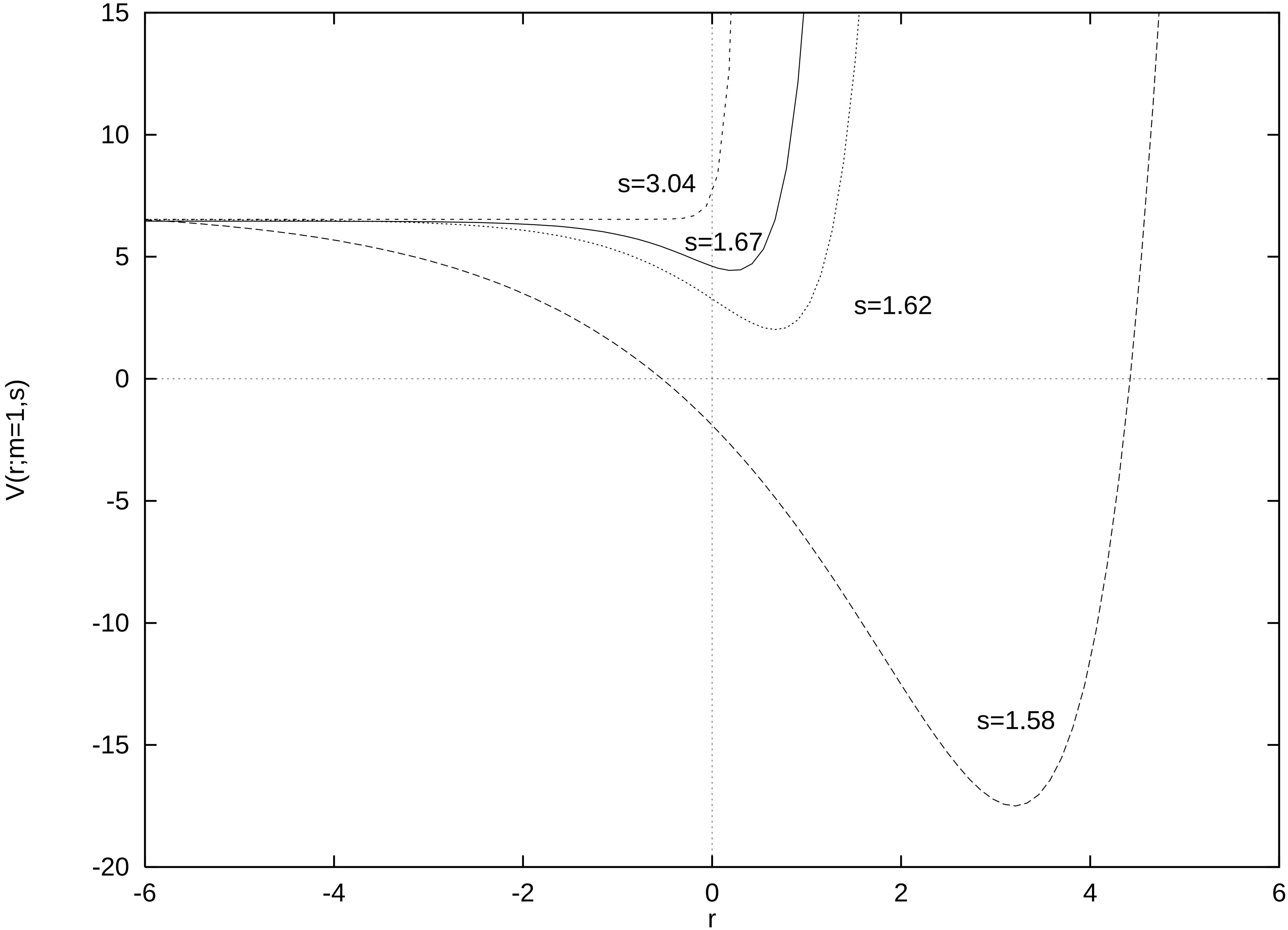
m=7/2
m=3
m=5/2
m=2
m=3/2
m=1
V(r;m,s=0.25)
r
40
30
20
10
0
-10
-20
0
1
2
3
4
5
6
7
8



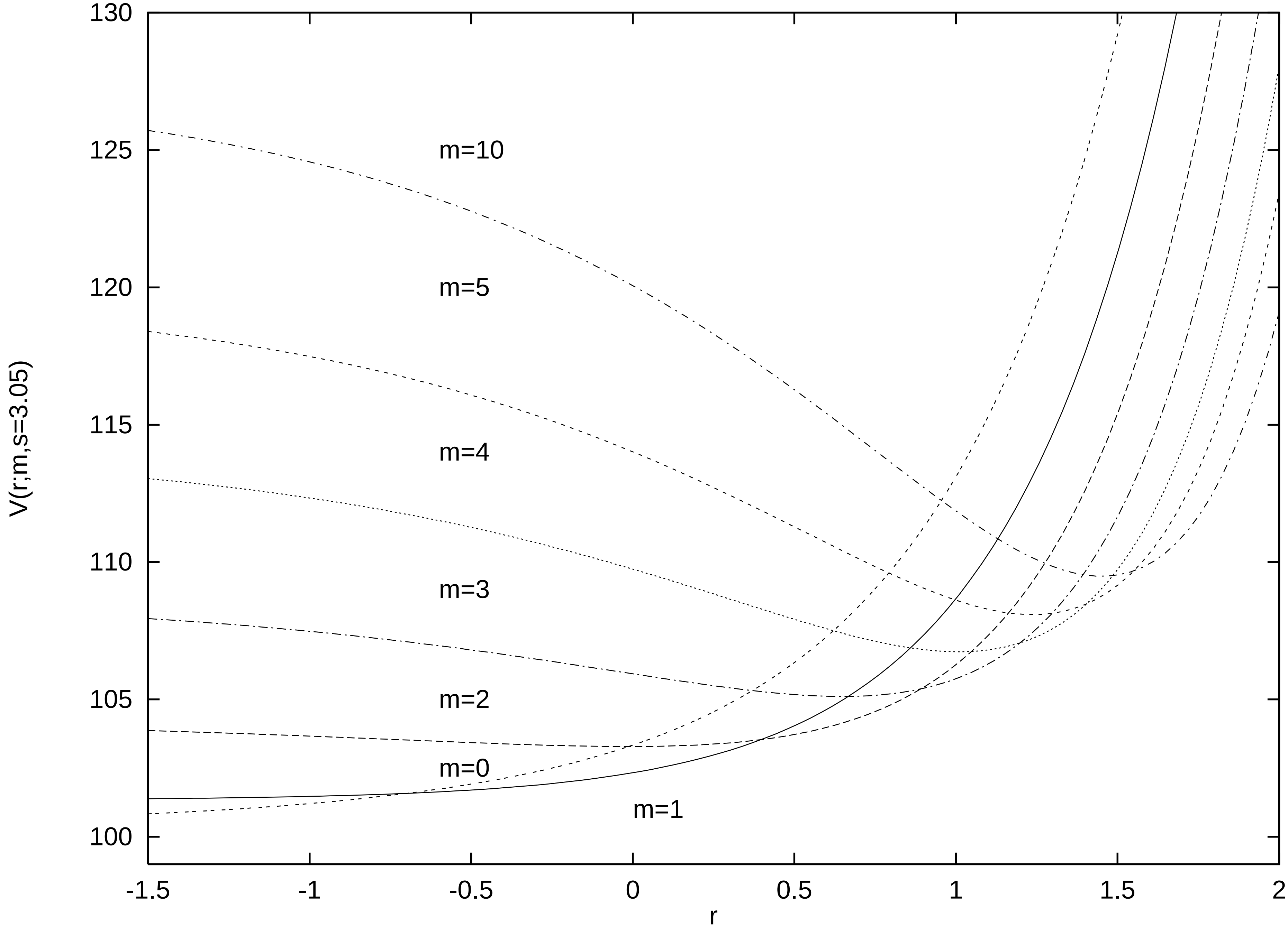
V(r;m=1,s)
r
s=3.1416
s=1.47
s=2
s=3.05 (V-100)
s=3 (V-50)
0
0.5
1
1.5
2
2.5
3
3.5
4
0
1
2
3
4
5



V(r;m,s=3)
r
m=3
m=2
m=1
m=1/2
m=3/2
m=5/2
60
58
56
54
52
50
48
46
44
-3
-2
-1
0
1
2



15
10
5
0
-5
-10
-15
-20
-6
-4
-2
0
2
4
6
r
V(r;m=1,s)
s=3.04
s=1.67
s=1.62
s=1.58


130
125
120
115
110
105
100
-1.5
-1
-0.5
0
0.5
1
1.5
2
r
V(r;m,s=3.05)
m=10
m=5
m=4
m=3
m=2
m=0
m=1